Math 321: Relations and functions, II

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- An expression describing what to do with the input, e.g. $f(x) = x^2 - 2$.
- A rule describing the output given the input, e.g. $\mathrm{lcm}(a, b)$ is the least common multiple of a and b.
- An algorithm describing how to compute the output from the input.

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What these all have in common is that it's about assigning outputs to inputs. That's the idea we'll use for defining functions in abstract generality.

Functions, formally

Let A and B be sets.

- A function from A to B is a set $f \subseteq A \times B$ of pairs $(a, f(a))$ so that for each $a \in A$ there is a unique $f(a) \in B$.
- We write $f : A \to B$.
- That is, we define a function as its graphs.
- Note that requiring the value $f(a)$ to be unique is saying the functino must satisfy the vertical line test.
- A is called the domain of f, also written $\operatorname{dom} f$, and B is called the codomain.
- The range of f, written $\operatorname{ran} f$, is the set of all $b \in B$ which are outputs $f(a)$ for some $a \in A$.

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In some contexts, it's more convenient to work with partial or multi-valued functions.

Functions, pictorally

Definitions with functions

Let $f : A \to B$ be a function.

- If the range of f is all of B, we say f is onto B or surjective onto B.
  If the codomain is clear, usually we just say onto or surjective.
- If $f(a) \neq f(a')$ whenever $a \neq a'$ are distinct inputs from A, we say f is one-to-one or injective.
  Equivalently, f is one-to-one if $f(a) = f(a')$ implies $a = a'$.
- If f is both one-to-one and onto B, we say f is a bijection onto B. This is also called a one-to-one correspondence between A and B.

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- The identity function on a set A is the function $\operatorname{id} : A \to A$ defined as $\operatorname{id}(a) = a$.

Some results

Theorem

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(Surjections) Homework :)

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A connection between functions and equivalence relations

Suppose you have a function $f : A \to B$.

- Define a relation $\sim_f$ on A as $x \sim_f y$ if $f(x) = f(y)$.
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Moreover, every equivalence relation arises like this from a function.

- Suppose $\sim$ is an equivalence relation on A.
- Define $f : A \to A/\sim$ as $f(x) = [x]$.
- Then, $\sim = \sim_f$.

Another connection between functions and equivalence relations

Suppose you have a set A, an equivalence relation $\sim$ on A, and a function $f : A \to A$.

- We say that f is well-defined on $\sim$-equivalence classes if $a \sim b$ implies $f(a) \sim f(b)$.
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- We need to see: if $a \equiv a' \mod 5$ and $b \equiv b' \mod 5$ then $a + b \equiv a' + b' \mod 5$ and $ab \equiv a'b' \mod 5$.

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These are calculations:

- We know $a - a' = 5k$ and $b - b' = 5\ell$ for some integers k, ℓ.

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- So $a + b = (a' + b') + 5(k + \ell)$, whence we conclude $a + b \equiv a' + b' \mod 5$.
- And $ab = (a' + 5k)(b' + 5\ell) = a'b' + 5b'k + 5a'\ell + 25k\ell = a'b' + 5(b'k + a'\ell + 5k\ell)$, whence we conclude $ab \equiv a'b' \mod 5$.

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More generally, addition and multiplication are well-defined modulo n for any $n > 1$, as you will prove in homework.

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- How many one-to-one functions are there from A to B?
- How many bijections are there from A to B?
- How many onto functions are there from A to B?