MATH 321: THE LEAST NUMBER PRINCIPLE AND INDUCTION

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The purpose of this note is to see that the least number principle and induction are two different looks at the same phenomenon. Let's begin by recalling just what these say. I'll go with the formulations in terms of sets.

Definition 1 (The Least Number Principle for $\mathbb{N}$)**.** If $X \subseteq \mathbb{N}$ is nonempty, then X has a least element.

Definition 2 (Induction)**.** If $X \subseteq \mathbb{N}$ has the property that, for each $n \in \mathbb{N}$, if $k \in X$ for all $k < n$ then $n \in X$, then $X = \mathbb{N}$.

For convenience, let's call $X \subseteq \mathbb{N}$ *inductive* if it satisfies the property you want to check for induction, i.e. if X satisfies that for each $n \in \mathbb{N}$, if every $k < n$ is in X then $n \in X$. We can then rephrase Induction as saying that if a subset of $\mathbb{N}$ is inductive then it must be all of $\mathbb{N}$.

Theorem 3. *The least number principle and induction are equivalent.*

Proof. (LNP $\Rightarrow$ Induction) Assume the least number principle, and we want to show that every inductive set must be $\mathbb{N}$. Suppose toward a contradiction that $X \subseteq \mathbb{N}$ is inductive but $X \neq \mathbb{N}$. Then, there must be some natural number n so that $n \notin X$. By the least number principle, there is a smallest such n. In particular, by the leastness of n we have that $k \in X$ for all $k < n$. Since X is inductive, we can conclude that $n \in X$. This is the desired contradiction.

(Induction $\Rightarrow$ LNP) Assume every inductive set is $\mathbb{N}$ and we want to prove that every nonempty subset of $\mathbb{N}$ has a least element. Suppose toward a contradiction that $X \subseteq \mathbb{N}$ is nonempty but doesn't have a least element. That is, if $n \in X$ then there is $k < n$ so that $k \in X$.

I claim that $Y = \mathbb{N} \setminus X$ must be inductive. To see this, fix an arbitrary $n \in \mathbb{N}$, and consider two cases. If $n \notin X$, then it must be that "if $k \in Y$ for all $k < n$, then $n \in Y$" is true, since any statement of the form "if P then Q" is true when Q is true. In the other case, if $n \in X$, then because X doesn't have a least element we have some $k < n$ so that $k \in X$. But then "if $k \in Y$ for all $k < n$, then $n \in Y$" is true, since it's of the form "if false then false". Either way, we get that Y satisfies the property for n, and so Y is inductive.

Since $Y = \mathbb{N} \setminus X$ is inductive, by induction it must be that $Y = \mathbb{N}$ and so $X = \emptyset$. But X was supposed to be nonempty. This is the desired contradiction. □

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