Math 321: Introduction

Kameryn J Williams

University of Hawai'i at Mānoa

Spring 2021

The scientific method

You probably learned this simplified picture for how scientists operate to produce knowledge.

1. Consider a natural phenomenon and make some preliminary observations.
2. Make a hypothesis about this phenomenon.
3. Design and carry out an experiment to test this hypothesis.
4. Analyze the results of the experiment.
5. Interpret this analysis to get a conclusion about the phenomenon.

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In mathematics, we produce knowledge differently.

The mathematical method

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A simplified picture of how mathematicians work:

1. Take an intuitive concept.
2. Mathematize the concept with a formal definition.
3. Make a conjecture about your definition.
4. Prove your conjecture, turning it into a theorem.
5. Conclude something about your concept.

An example of how to math: Alan Turing

Alan Turing

- Studied the concept of computability—what does it mean to compute an answer to a problem?
- Formalized this concept—now called the Turing machine.
- Proved there are some problems a Turing machine cannot compute the answer to.
- Conclusion: there are some problems a computer cannot compute an answer to.

The goal of this class

The goal of this class is to teach you the mathematical method so that you can go on to use it in future math classes. In lecture, your professors will state and prove theorems, and you need to be able to understand the underlying method to follow along. And for homework you will be asked to prove things.

The main focus is proofs. You should learn three things:

- How to read and understand proofs.
- How to construct your own proofs.
- How to write up a proof so that other mathematicians can understand your ideas.

We will also learn some basic language of mathematics—some concepts that appear over and over in different areas of math.

The purpose of proofs

Proofs do multiple things for us.

- They help us determine what is true.
- They help us determine why things are true.
- We can extract extra information by looking more closely at a proof.

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$$d = \sqrt{s^2 + s^2} = \sqrt{2}\, s.$$

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- So what we're really asking is: is $\sqrt{2}$ a rational number?

It's not

Theorem

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  - A number x is rational if is a ratio of two integers: $x = p/q$ for $p, q \in \mathbb{Z}$.
  - A number is irrational if it is not rational.

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- But how do we show this cannot happen?

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- But how do we show this cannot happen?

Our strategy, known as proof by contradiction or reductio ad absurdum, will be to show that it is impossible for $\sqrt{2}$ to be a ratio of two integers. That is, we suppose it were the case that $\sqrt{2}$ is rational and reason from there to show that can't actually happen.

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- So p and q are both even. But also they have no common divosor. So this is impossible!

Communicating mathematical arguments

We step by step reasoned out why $\sqrt{2}$ must be irrational. But when writing up mathematical arguments to be read by others, we write in complete sentences, paragraphs, etc., with mathematical formulae mixed in as appropriate.

Let's see an example of how this looks.

Writing up the proof

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Proof.

Suppose toward a contradiction[a] that $\sqrt{2}$ is rational. Then, we have that $\sqrt{2} = p/q$ for some integers p and q with no common divisors. Doing a bit of algebra yields $p^2 = 2q^2$, whence we conclude that p^2 is even. So p also must be even, and we can write it as $p = 2k$ for some integer k. Substituting this in gives us $q^2 = 2k^2$. That is, q^2 and hence also q must be even. Since p and q are both even, they have 2 as a common factor. But this contradicts that p and q have no common factors. So our assumption that $\sqrt{2}$ is rational must be impossible, and we conclude that $\sqrt{2}$ is irrational. □

[a] This verbiage tells the reader that we're doing a proof by contradiction.

A question for you to think about

Can you generalize this way of thinking to see why $\sqrt{3}$ must be irrational? What about $\sqrt{n}$ for any positive integer n? When is it irrational?