

Math 321: Infinity, I

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Hilbert's hotel

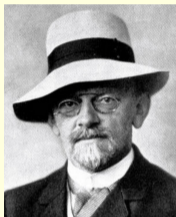
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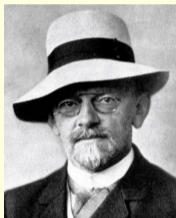
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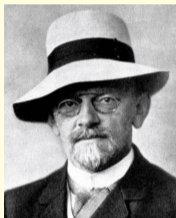


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What if instead of just you, you show up with 1000 friends? Can he still fit you all in (without making you share a room)?

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- An infinite train appears. There are cabs n for each $n \in \mathbb{N}$ and each cab has seats s for each $s \in \mathbb{N}$. Can you fit them all in?

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- An infinite train appears. There are cabs n for each $n \in \mathbb{N}$ and each cab has seats s for each $s \in \mathbb{N}$. Can you fit them all in?
- A half-marathon ends at the hotel and the runners, densely packed with one runner for each positive rational number, each need rooms. Can you fit them all in?

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Before we answer this, let's make precise the mathematical ideas we've been playing with.

Let's make this mathematically precise

Definition

A set X is **countable** if there is a one-to-one function $r : X \rightarrow \mathbb{N}$.

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Define h by cases: if $x \in A$, then $h(x) = 2 \cdot f(x)$, and if $x \in B \setminus A$ then $h(x) = 2 \cdot g(x) + 1$. That is, put A in the even rooms and put whatever remains of B in the odd rooms. It's clear h is one-to-one.

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Let's be a bit more precise about enumerations

- An **enumeration** is a listing of elements index by natural numbers: x_0, x_1, \dots
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- For example, we can enumerate \mathbb{Z} as:

$$0, 1, -1, 2, -2, 3, -3, \dots$$

So \mathbb{Z} is countable.

More countable sets

Theorem

$\mathbb{N} \times \mathbb{N}$ is countable.

Proof 1: Define $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ as $f(a, b) = 3^a 5^b$. This is one-to-one by the fundamental theorem of arithmetic.

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The previous diagonals give $1 + 2 + \cdots + (x + y)$ points. As we proved a few chapters ago, this sum is equal to $(x + y)(x + y + 1)/2$. And (x, y) is the $(y + 1)$ -th point on its diagonal.

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Corollary

\mathbb{Q} is countable.

We can think of a rational number p/q written in simplest form as a pair (p, q) of integers. Since $\mathbb{N} \times \mathbb{N}$ and \mathbb{Z} are both countable we have $\mathbb{Z} \times \mathbb{Z}$ and thus also \mathbb{Q} are countable.

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Composing f with the bijection $p : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ gives a one-to-one function from A to \mathbb{N} , showing that A is countable.

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Let's construct a one-to-one function $f : \mathbb{N}^* \rightarrow \mathbb{N}$. Send the empty sequence with zero elements to 0. Given a nonempty sequence $\vec{s} = s_0, s_1, \dots, s_n$ of natural numbers set

$$f(\vec{s}) = 2^{s_0+1} \dots 3^{s_1+1} \dots p_n^{s_n+1},$$

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We can use an enumeration of A to translate the results about $\mathbb{N} \times \mathbb{N}$, \mathbb{N}^k , and \mathbb{N}^* to A .

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We've been seeing a lot of examples of countable sets.

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- \mathbb{Z}
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- The set of possible computer programs.
- The set of possible books written in the English language.

- The set of polynomials with rational coefficients.
- Hence also the set of **algebraic numbers**—numbers which are roots of polynomials with rational coefficients.

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- A^* , the set of finite sequences from A , for any countable set A

This last one implies a lot of sets are countable:

- The set of possible computer programs.
- The set of possible books written in the English language.

- The set of polynomials with rational coefficients.
- Hence also the set of **algebraic numbers**—numbers which are roots of polynomials with rational coefficients.

Is there any set which is not countable????

Having asked this, let's return to Cantor's cruise ship.

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There is a small detail to be addressed.

- Some numbers have *two* decimal expansions.
- For example, $1 = 1.000\dots = 0.999\dots$
- In general, if you can write a number so its decimal expansion is eventually 0s, you could instead write it to eventually be all 9s.

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Because d only has 4s and 5s, it avoids this issue. It has a unique decimal expansion.