Math 321: Infinity, I

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Hilbert's hotel

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He tells you that the hotel is completely booked. Room n is filled already for every room number $n \in \mathbb{N}$.

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How?

What if instead of just you, you show up with 1000 friends? Can he still fit you all in (without making you share a room)?

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- An infinite train appears. There are cabs n for each $n \in \mathbb{N}$ and each cab has seats s for each $s \in \mathbb{N}$. Can you fit them all in?
- A half-marathon ends at the hotel and the runners, densely packed with one runner for each positive rational number, each need rooms. Can you fit them all in?

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Before we answer this, let's make precise the mathematical ideas we've been playing with.

Let's make this mathematically precise

Definition

A set X is countable if there is a one-to-one function $r : X \to \mathbb{N}$.

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Define h by cases: if $x \in A$, then $h(x) = 2 \cdot f(x)$, and if $x \in B \setminus A$ then $h(x) = 2 \cdot g(x) + 1$. That is, put A in the even rooms and put whatever remains of B in the odd rooms. It's clear h is one-to-one.

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Alternatively, we can think of the proof in terms of enumerations: A and B are both countable, so can enumerate their elements as $a_0, a_1, \cdots, a_n, \cdots$ and $b_0, b_1, \cdots, b_n, \cdots$

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Let's be a bit more precise about enumerations

- An enumeration is a listing of elements index by natural numbers: $x_0, x_1, \ldots$
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- For example, we can enumerate $\mathbb{Z}$ as:

$$0, 1, -1, 2, -2, 3, -3, \cdots$$

So $\mathbb{Z}$ is countable.

More countable sets

Theorem

$\mathbb{N} \times \mathbb{N}$ *is countable.*

Proof 1: Define $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ as $f(a, b) = 3^a 5^b$. This is one-to-one by the fundamental theorem of arithmetic.

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The previous diagonals give $1 + 2 + \cdots + (x + y)$ points. As we proved a few chapters ago, this sum is equal to $(x + y)(x + y + 1)/2$. And (x, y) is the $(y + 1)$-th point on its diagonal.

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Corollary

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Corollary

$\mathbb{Q}$ *is countable.*

We can think of a rational number p/q written in simplest form as a pair (p, q) of integers. Since $\mathbb{N} \times \mathbb{N}$ and $\mathbb{Z}$ are both countable we have $\mathbb{Z} \times \mathbb{Z}$ and thus also $\mathbb{Q}$ are countable.

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Composing f with the bijection $p : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ gives a one-to-one function from A to $\mathbb{N}$, showing that A is countable.

The countable sets don't stop coming

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Let's construct a one-to-one function $f : \mathbb{N}^* \to \mathbb{N}$. Send the empty sequence with zero elements to 0. Given a nonempty sequence $\vec{s} = s_0, s_1, \ldots, s_n$ of natural numbers set

$$f(\vec{s}) = 2^{s_0+1} \cdots 3^{s_1+1} \cdots p_n^{s_n+1},$$

i.e. the product of the i-th prime p_i to the power $s_i + 1$, for $i \leq n$. This map is one-to-one by the fundamental theorem of arithmetic.

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We can use an enumeration of A to translate the results about $\mathbb{N} \times \mathbb{N}$, $\mathbb{N}^k$, and $\mathbb{N}^*$ to A.

Is there anything that isn't countable?

We've been seeing a lot of examples of countable sets.

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- $\mathbb{Z}$
- $\mathbb{Q}$
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- The set of possible computer programs.
- The set of possible books written in the English language.
- The set of polynomials with rational coefficients.
- Hence also the set of algebraic numbers—numbers which are roots of polynomials with rational coefficients.

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Having asked this, let's return to Cantor's cruise ship.

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We'll prove this by showing that given any enumeration $x_0, x_1, \ldots$ of real numbers there is a real number d which is not part of the enumeration.

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We'll prove this by showing that given any enumeration $x_0, x_1, \ldots$ of real numbers there is a real number d which is not part of the enumeration.

Define d to be the real number between 0 and 1 with the decimal expansion

$$0.d_0 d_1 d_2 \cdots$$

given by the rule: $d_n = 5$ if the n-th digit of x_n is 4, else $d_n = 4$.

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There is a small detail to be addressed.

- Some numbers have *two* decimal expansions.
- For example, $1 = 1.000\ldots = 0.999\ldots$.
- In general, if you can write a number so its decimal expansion is eventually 0s, you could instead write it to eventually be all 9s.

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Because d only has 4s and 5s, it avoids this issue. It has a unique decimal expansion.