Math 321: A little more induction

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Proofs by induction

Induction is a method to prove something is true for all natural numbers.

Common Induction

- (Base case) Show $P(0)$;
- (Inductive step) Consider an arbitrary n. Assume $P(n)$, show $P(n+1)$,

Strong Induction

- Consider an arbitrary n. Assume $P(k)$ for all $k < n$, show $P(n)$.

Nested Induction

An inductive argument can be part of a larger proof. You can even have an inductive argument inside a larger inductive argument!

- Suppose you want to prove a statement of the form "for all natural numbers n, and m, they satisfy some property $P(n, m)$".
- You might try to prove this by induction on n.
  - That is, you prove that $P(0, m)$ is true for all m; and
  - You assume $P(n, m)$ is true for all m and you show $P(n + 1, m)$ is true for all m.
- Those two steps are themselves statements about all natural numbers m, and you might prove them by induction on m.

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Let's see an example.

Buckets of fish

Let's imagine we go down to the beach toplay a game, call it buckets of fish.

- The setup is, there are finitely many buckets in a row, each bucket contains a finite number of fish, and we have a large supply of extra fish to use as needed.
- There are two players. On your turn, you remove a fish from one bucket, and then adding as many extra fish as you want to each of the buckets to the left.
- For example, you could take a fish from bucket 3, and then put 3 fish in bucket 0, no fish in bucket 1, and 2000 fish in bucket 2.
- The winner is whoever takes the very last fish.

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Must this game ever end? Or could we possibly collaborate to play buckets of fish forever?

A warm-up

Let's first see that any player can always force the game to end after finitely many moves.

Theorem

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Theorem

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Proof.

No fish can ever be added to the rightmost bucket. So if you always grab a fish from the rightmost bucket, you'll reduce its number by 1 with no way of increasing. Continually play with this strategy and the game will eventually end, no matter what your opponent does. □

The solution

Theorem

Any play of buckets of fish ends after finitely many moves, no matter how the two players play.

Proof.

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So focus on the inductive step. Assume that any game with n buckets always ends after finitely many steps, and consider a game with $n + 1$ many buckets, where the last bucket has k many fish.

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So focus on the inductive step. Assume that any game with n buckets always ends after finitely many steps, and consider a game with $n + 1$ many buckets, where the last bucket has k many fish.

We will prove this game must always end by induction on k. □

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Theorem

In game of buckets of fish with $n+1$ many buckets and k fish in the last bucket, the game always ends no matter how the players play.

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Proof (Base case).

The base case $k = 1$ isn't too hard. If a player ever takes the one fish in the last bucket, then the game becomes an n bucket game, which we know must end by inductive hypothesis. If both players ignore the last bucket as long as they can, they're essentially playing an n bucket game. It must end after finitely many steps, so eventually someone is forced to empty the last bucket. □

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A similar idea gives the inductive step.

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In game of buckets of fish with $n+1$ many buckets and k fish in the last bucket, the game always ends no matter how the players play.

Proof (Inductive step).

Suppose that any $n+1$ bucket game where the last bucket has k fish game is forced to end. Consider an $n+1$ bucket game where the last bucket has $k+1$ many fish.

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Proof (Inductive step).

Suppose that any $n+1$ bucket game where the last bucket has k fish game is forced to end. Consider an $n+1$ bucket game where the last bucket has $k+1$ many fish.

If a player ever takes a fish from the last bucket, it becomes an $n+1$ bucket game where the last bucket has k fish. By inductive hypothesis, we know such a game must eventually end.

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Proof (Inductive step).

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If a player ever takes a fish from the last bucket, it becomes an $n+1$ bucket game where the last bucket has k fish. By inductive hypothesis, we know such a game must eventually end. The two players can try to avoid taking a fish from the last bucket as long as possible. But this is in effect playing an n bucket game, which must end after finitely many steps. So eventually someone must take a fish from the last bucket. □