MATH 321: IN-CLASS WORKSHEET 6
FRIDAY, FEBRUARY 19TH

Today's worksheet is about using proofs by induction to establish facts about how to represent natural numbers.

The standard way we represent numbers is with base-10. We can represent a number as a finite sequence of digits 0 through 9, which each digit's place represents a power of 10: the rightmost digit is the unit's place, the next is the ten's place, then the hundred's place, and so on. But there's nothing special about 10 here. You can instead represent numbers in base-b, where $b > 1$ is an integer. For base-b, you use the digits 0 through $b - 1$, and each place represents a power of b.[1] For example, in base-3, the sequence $1201 = 1 \cdot 3^0 + 0 \cdot 3^1 + 2 \cdot 3^2 + 1 \cdot 3^3$ represents the number we write in base-10 as 46. If you instead wanted to write 46 in base-2, that would be $101110 = 0 \cdot 2^0 + 1 \cdot 2^1 + 1 \cdot 2^2 + 1 \cdot 2^3 + 0 \cdot 2^4 + 1 \cdot 2^5$. Just like in base-10, we write the smallest power on the right and the largest power on the left.

(1) Determine how to write the number 79 (here this is base-10) in base-2.
(2) Look at Theorem 30 and Corollary 31 on page 34 of the textbook, where the author proves that every natural number has a unique base-2 representation. Read through the proofs and understand how strong induction is used here.
(3) Determine how to write the number 79 (this is base-10 again) in base-3.
(4) Build on the proof that every natural number has a unique base-2 representation to show that every natural number has a unique base-3 representation (Exercise 4.11).
    - For the base-2 proof, an important fact was that if $n > 0$ then there is a largest power 2^k of 2 so that $2^k \leq n$. This is also true for 3 instead of 2, but there's an extra catch: it might be that not only is $3^k \leq n$, but also $2 \cdot 3^k \leq n$. (For instance, 3^3 is the largest power of 3 which is ≤ 79, but also $2 \cdot 3^3 \leq 79$. On the other hand, 3^3 is the largest power of 3 which is ≤ 33, but $2 \cdot 3^3 > 33$.) A question you should ask: why can't it be that also $3 \cdot 3^k \leq n$ in this case?
    - You don't want to represent a number as a sum of powers of 3, but rather you want to multiply each power of 3 by either 1 or 2. (Or multiply by 0, but that's the same as not including it.)
(5) Time permitting, generalize your proof that every number has a unique base-3 representation to show that every number has a unique base-10 representation.
(6) Time permitting, how far can you generalize this? Can you show, for any integer $b > 1$, that every natural number has a unique base-b representation?

There is nothing to submit for today's worksheet, but (4) is one of your homework problems due this week.

[1] If you're actually writing these, not just working abstractly, when $b > 10$ there's the question of what to write for each digit. For instance, if you're using base-16 you don't want to just write 11 for a digit, since how do you tell whether that's the digit 11 or the digit 1 followed by the digit 1? In practice, the solution is to introduce new characters for the extra digits. For base-16, also known as *hexadecimal*, the convention is to use letters from the alphabet: `a` is the digit for 10, `b` for 11, `c` for 12, `d` for 13, `e` for 14, and `f` for 15. So in hexadecimal,

$$\texttt{b0fa} = 10 \cdot 16^0 + 15 \cdot 16^1 + 0 \cdot 16^2 + 11 \cdot 16^3$$

is the number you would write in base 10 as 45306.