Math 321: Graph Theory

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The bridgens of Königsberg

A challenge: Can you take a stroll through town and cross each bridge exactly once?

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For the purposes of this class, the answers are: Yes, Yes, Yes.

Graphs and relations

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You can think of graphs, in the general multi-edge context, as generalizing relations. Not only can we say whether two points are related, we can have them related in multiple different ways.

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The Königsberg bridge question, stated in this language, becomes: Is there a circuit (or path) through the Königsberg graph which uses each edge exactly once.

- Call such a circuit (path) a Eulerian circuit (Eulerian path).

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- The vertices in the Königsberg graph have degrees 3, 3, 3, and 5, so it doesn't admit a Eulerian circuit.

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The circuit has to visit v, since the graph is connected. And when it visits v it comes in via an edge and then out via another edge. So each time the circuit visits v gives a pair of edges touching v. Thus, v has even degree.

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When you visit a new vertex, you then leave it, using up two of its edges. Since each vertex has even degree, the only way you can get stuck is to arrive at a vertex with only an odd number of unused edges. But the only such vertex is v_0 itself.

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If this circuit is an Eulerian circuit, we are done. Otherwise, we need to modify it with a detour.

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There's some vertex with an unused edge, since by assumption our circuit is not Eulerian. If all vertices in our circuit used all edges, then this would mean we have a vertex which isn't reachable from the vertices in our circuit, contradicting that the graph is connected.

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Now fix this vertex v_1. Repeating the process, travel along v_1, only using unused edges, until you get back to v_1. Just as before, eventually you get back to where you started.

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Now fix this vertex v_1. Repeating the process, travel along v_1, only using unused edges, until you get back to v_1. Just as before, eventually you get back to where you started.

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Because the graph is finite, we eventually run out of unused edges.

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Starting at v_0 and taking all detours, we get an Eulerian circuit. □

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As in the circuit case, any middle node in the path must have even degree, since we can pair up edges in/out. So the only possible odd degrees occur at the start/end vertices. And if all nodes have even degree then, like in the circuit case, the starting vertex must be the ending vertex.

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Consider a new graph, with the same vertices, where we add a new edge between s and e.

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Consider a new graph, with the same vertices, where we add a new edge between s and e.

In this larger graph, every vertex has even degree. So it admits an Eulerian circuit. Now observe that if we remove the s to e edge from the circuit, we get an Eulerian path for the original graph. Done. □

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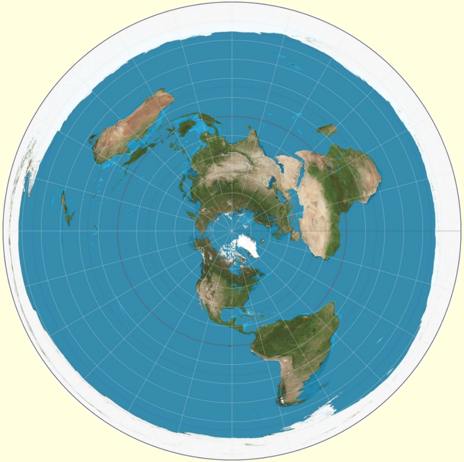

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Drawn in the plane, the faces of the polyhedron correspond to regions, including the unbounded outside region.

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$$v - e + f,$$

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The base case is 0 edges and 1 vertex. So there's one facing, giving $v - e + f = 2$.

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And these are the only two ways to add a new edge, so we're done!