Math 321: The theory of games

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An example

Let's play the game Twenty-One. There's two players, who take turns counting up to twenty-one, starting at one. On each turn you can say the next one, two, or three numbers, no fewer and no more. The winner is whomever says twenty-one.

Twenty-One

Theorem

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Proof.

Here's the strategy: you want to end your turn on 1, 5, 9, 13, 17, or ultimately 21. If you end your turn on one of these, then your opponent can only add 1, 2, or 3 to it, allowing to respond by ending on the next number on the list. Since you can end on 1 for your first turn, this means you can put yourself in a winning position and then win by ensuring you never leave this position. □

A generalized example

Let's generalize. Instead of counting up by up to 3 with a goal of 21, we could make the step size and goal any natural numbers. Let's call the game with goal n and step size s as $G(n, s)$, so Twenty-One was $G(21, 3)$.

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The winning startegy, in either case, is the same: you want to end your turn on the numbers with the same remainder r as n divided by $s + 1$.

$$r, s + 1 + r, 2s + 2 + r, \ldots, n$$

Because these are spaced out by exactly $s + 1$, no matter what move your opponent plays you can respond to stay in your winning position.

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If n is a multiple of $s+1$, this remainder is 0, so the first player cannot end their first turn in a winning position, so it is the second player who can force to be in a winning position. Otherwise, the remainder is $1 \leq r \leq s$, so the first player can get in a winning position on the first move. □

Buckets of fish

Recall the buckets of fish game we talked about earlier as part of an example of an inductive proof. There are finitely many buckets arranged in a row, and each starts with some finite number of fish. Each turn, a player removes a fish from one bucket and puts as many new fish as they like in any of the buckets to its left. The winner is whomever takes the last fish.

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We saw that any game of buckets of fish must eventually end, but must it be that one of the two players has a winning strategy? Or could it be that for every game each player has a shot at winning no matter how their opponent plays?

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Observe that after you take the last fish, all buckets have 0 fish, an even number for each. Next, notice that if there's a bucket with an odd number of fish, then you can take one fish from the right-most odd bucket, making it even, and add fish to more left odd buckets to make them odd. So you can get back to the winning position. On the other hand, when your opponent faces an all even setup, because they have to take only one fish from a bucket, they make it odd, keeping them out of the winning position.

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Can we generalize?

- In all the examples we've seen so far, it was always the case that someone has a winning strategy. (And the textbook has more examples of games with winning strategies.) Can we generalize this? What sort of general statement can we make about when a game has a winning strategy?

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- (Games with > 2 players) Consider the game with three players A, B, and C, which has one inning: Player A decides which of B or C wins, and then the game ends. No one has a winning strategy here.
- (Games with draws) In tic-tac-toe any player can force a draw, so there is no winning strategy. Or a simpler example: the game where each player does nothing and then it ends in a draw.

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- (Games with randomness) In general, there's no way to handle this. You can investigate strategies that have high probability of winning, and this is an area of ongoing mathematical investigation. but that's taking us away from the question of a guaranteed winning strategy—and also it gets really hard fast—so let's not consider these.

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- (Games with > 2 players) These are also hopeless for a general theory, so let's exclude these two.
- (Games with draws) Here, maybe we can amend things to say that either a player has a winning strategy or everyone can force a draw.

Abstract games

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We will consider both games with draws and games without.

Game Trees

When thinking about games in abstract generality, it turns out to be useful to think of them as trees. A position in the game can be thought of as the sequence of legal moves which led up to it. We can then order the positions in a tree—a position is below another if it's a longer sequence of moves—and this tree represents every possible play of the game.

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Because the game is finite, this tree has no infinite paths. It might have infinite width, but any play through the game eventually stops at some terminal position, which is either winning for one player or else is a draw.

An example game tree: ~~Twenty-One~~ Seven

The fundamental theorem of finite games

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- If it is Achilles's turn, we look to see if there is a position he can play to which we have labeled with A. If so, we label the position with A. Else, if every next position is labeled with P, we label the position with P.
- If it is Patroclus's turn, we do the same but backward. If there is a position he can play to labeled P, we label the current position with P, else we label it with A.

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This is a form of induction on the tree.

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Proof.

If the starting position is labeled A, then Achilles has a winning strategy: he always plays to an A position, which is possible by the recursive construction of the labels. Then, on Patroclus's turn, he has no choice but to play to an A position, so Achilles stays in a winning position, ensuring his eventual victory.

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For the other possibility, if the starting position is labeled P, then Achilles has no choice but to play to a P position. Patroclus can play to stay in a P position, and Achilles will never have a chance to break Patroclus out of his winning position. □

What about games with draws?

Using the ideas from Zermelo's proof, we can also answer the case with draws. We'll do this in two ways.

- We modify the proof, allowing for draws.
- We derive it as a corollary from the theorem without draws.

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- Label each terminal position with A, D, or P, depending on whether it is a win for Achilles, a draw, or a win for Patroclus.

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- Label each terminal position with A, D, or P, depending on whether it is a win for Achilles, a draw, or a win for Patroclus.
- If a non-terminal position is at Achilles's turn: If Achilles can play to an A-labeled position, label the current position with A.
  - Else if Achilles can play to a D-labeled position, label it D.
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  - Else if Achilles can play to a D-labeled position, label it D.
  - Else, label it P.
- Patroclus's turns are handled similarly: If he can play to a P-labeled position, label it P. Else if he can play to a D-labeled position, label it D. Else, label it A.

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- If the starting position is labeled P, then Achilles must play to a P condition. But then Patroclus can play to keep it labeled P, and Achilles can never escape from P labels.
- If the starting position is labeled D, then Achilles cannot play to an A position. But he can play to a D position. And on Patroclus's turns, he's in a symmetric situation. He cannot play to a P position, but he can play to a D position. So if both always play to a D position, the game will eventually end in a draw.

Games with draws, as a corollary to games without draws

Consider a game G allowing draws. Let's define two new games, which don't allow draws.

- G_A is the game G, except that draws are counted as wins for Achilles.
- G_P is the game G, except that draws are counted as wins for Patroclus.

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Observe that it cannot be that Achilles has a winning strategy for G_P while Patroclus has a winning strategy for G_A. So that leaves three possibilities.

Games with draws, as a corollary to games without draws

- Case 1: Achilles has a winning strategy for both G_A and G_P. Note that his winning strategy for G_P is also a winning strategy for G, since the only way he can win G_P is if he would've won G.

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- Case 1: Achilles has a winning strategy for both G_A and G_P. Note that his winning strategy for G_P is also a winning strategy for G, since the only way he can win G_P is if he would've won G.
- Case 2: Patroclus has a winning strategy for both G_A and G_P. Then he has a winning strategy for G, by playing according to the strategy for G_A.

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- Case 1: Achilles has a winning strategy for both G_A and G_P. Note that his winning strategy for G_P is also a winning strategy for G, since the only way he can win G_P is if he would've won G.
- Case 2: Patroclus has a winning strategy for both G_A and G_P. Then he has a winning strategy for G, by playing according to the strategy for G_A.
- Case 3: Achilles has a winning strategy for G_A and Patroclus has a winning strategy for G_P. So if they both play according to these winning strategies, the only possibility is that G ends in a draw.

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- Chess, when played according to tournament rules, is a finite game. (There are rules to ensure the game doesn't last forever.) But it is currently not known who has the winning strategy or whether it's forced-draw strategies.
- Go. Played on a 5×5 board, there is a known algorithm for the first player to win. For the 19×19 board used in usual play, it is still an open question to find a winning or forced-draw strategy.

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Trying to generalize this as far as possible takes us to the cutting edge of mathematical research, so we won't do so in this class.