Math 321: Some discrete mathematics

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Spring 2021

Discrete Mathematics

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- Combinatorics
- Graph theory
- Number theory
- Some stuff in theoretical computer science
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Let's see some examples.

Pointing at people

Imagine the following situation.

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- The group decides that everyone will point at each other. You can point at any number of people, including yourself, and you can point at someone multiple times.

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- You are at a party with a large (but finite!) group of people.
- The group decides that everyone will point at each other. You can point at any number of people, including yourself, and you can point at someone multiple times.
- The question: Can you arrange things so that everyone is pointed at more often than pointing to?

This may seem like a frivolous situation, and the framing certainly is. But there is a real mathematical structure lurking here. In jargon the question is: Given a finite directed graph, is it possible for every node in the graph to have larger in-degree than out-degree?

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For each person, let their pointing number be the number of people they are pointing at, and their receiving number be the number of people they are pointing to. Let P be the sum of the pointing numbers and R be the sum of the receiving numbers.

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For each person, let their pointing number be the number of people they are pointing at, and their receiving number be the number of people they are pointing to. Let P be the sum of the pointing numbers and R be the sum of the receiving numbers. I claim that $P = R$. This is because each instance of pointing adds 1 each to P and R; if I point at you, that's $+1$ to my pointing number and $+1$ to your receiving number.

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Here's the setup. Everyone will give \$1 to each person they are pointing at. Since everyone is pointed at more than they point, this means everyone makes a profit. And we can repeat this as much as we like to get everyone as much money as we want.

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Graph theory

As was mentioned earlier, this is an example of a theorem from graph theory, a branch of discrete math that sees a lot of application in computer science.

We'll dig deeper into graph theory after spring break. But for now, let's do a little bit of combinatorics.

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Let's answer the question where we count them differently—that is, where order matters.

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Imagine you place down n many 1s in a row: $1\ 1\ 1\ \cdots\ 1\ 1$.

In the spaces between the 1s we will either place a $+$ or leave it blank. We can then interpret that as a sum, grouping together a contiguous block into one number. And observe that any sum for n can be represented this way.

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The number of permutations for a list depends only upon the number of objects in the list. Just what number is it?

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The answer depends on whether we care about order.

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Imagine picking the objects one at a time, until you've picked all k. How many choices are there for the first one? The second? And so on?
This gives a total of

$$n(n-1)\cdots(n-k+1)$$

many ways to pick the objects. Now observe that

$$\frac{n!}{(n-k)!} = \frac{n(n-1)\cdots(n-k)(n-k-1)\cdots 1}{(n-k)(n-k-1)\cdots 1} = n(n-1)\cdots(n-k+1).$$

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