MATH 302: WEEK 8 WORKSHEET

Today's worksheet is about simple harmonic motion. An object is said to move according to *simple harmonic motion* if its position x as a function of time t behaves according to the second-order differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \omega^2 x = 0,$$

where ω is a constant.

(1) Find the general solution to the simple harmonic motion differential equation, written in terms of the sin and cos functions. What role does the constant ω play in the solution?

(2) The form of the solution you just calculated usually isn't the most useful for applications, so let's see how to transform it to a more useful form. Show that, if c_1, c_2, ω are constants, then

$$c_1 \sin(\omega t) + c_2 \cos(\omega t) = c \sin(\omega t + \varphi),$$

for some new constants c and φ, by following these steps:

- Draw a right triangle with side lengths c_1, c_2, and $\sqrt{c_1^2 + c_2^2}$ where one of the angles is φ.[1] Use this triangle to determine what $\sin\varphi$ and $\cos\varphi$ are in terms of c_1 and c_2.
- Next, rewrite the expression as

$$c_1 \sin(\omega t) + c_2 \cos(\omega t) = \sqrt{c_1^2 + c_2^2}\left(\frac{c_1}{\sqrt{c_1^2 + c_2^2}}\sin(\omega t) + \frac{c_2}{\sqrt{c_1^2 + c_2^2}}\cos(\omega t)\right).$$

  Substitute $\sin\varphi$ and $\cos\varphi$ into this expression, using what you calculated for their values.
- Use the angle sum identity $\sin\alpha\cos\beta + \cos\alpha\sin\beta = \sin(\alpha+\beta)$ to rewrite your expression into the desired form $c\sin(\omega t + \varphi)$. What is the value of the constant c?

(3) Hooke's law states that if you have a weight hanging from an ideal spring (discounting friction and similar) then the force the spring imparts on the weight is proportional to the weight's distance from its equilibrium position. In symbols,

$$F = -kx,$$

where $k > 0$ is a constant that depends on the spring. Using the fact that acceleration from a force in the force divided by the objects mass, write and solve a differential equation which describes the position of the object as a function of time. [Hint: this question is part of the problem set about simple harmonic motion for a reason. ⌣̈]

(4) A weight is hung from an ideal spring, and let come to equilibrium. You then pull down the weight 1 meter from its equilibrium position, hold it unmoving with an initial velocity of 0 meters per second and let go, setting it in motion. You then measure that it takes 3 seconds for the weight to return to its equilibrium position. Determine the weights position x as a function of time t. How long is it until the weight returns to the position it started at?

[1] There's two angles you could pick to be φ. One choice will lead you to the expression $c\sin(\omega t + \varphi)$ while the other will lead you to showing that you could instead rewrite $c_1 \sin(\omega t) + c_2 \cos(\omega t)$ to be in the form $c\cos(\omega t - \varphi)$. I only ask you to show one of the two. If your work is leading you to the version with cos, try switching which angle you called φ.