MATH 302: WEEK 7 WORKSHEET

Today's worksheet is about checking when sets of functions are linearly independent. We will focus here on looking at pairs of functions at a time. Let me recall for you the definition in this context: two functions $f(x)$ and $g(x)$ are *linearly dependent* if there are constants c and d, not both 0, so that $cf(x) + dg(x) = 0$. (Here this is equality of functions, i.e. the lefthand side is 0 for any input x.) If they are not linearly dependent, we say they are *linearly independent*.

One tool for checking linear independence, which also has other uses in differential equations, is the Wronskian.[1] If f and g are functions of one variable, then their *Wroskian*[2] is:

$$W(f, g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - gf'.$$

The following theorem gives a necessary, but not a sufficient, condition for two functions to be linearly dependent. Phrased in the contrapositive, this gives a sufficient, but not necessary, condition for two functions to be linearly independent.

Theorem. *Suppose f and g are differentiable functions. If they are linearly dependent, then their Wroskian $W(f, g)$ is identically 0. Phrased in the contrapositive, if their Wronskian is not identically 0, then they are linearly independent.*

Problem.

(1) Consider the functions $f(x) = e^{ax}$ and $g(x) = e^{bx}$, where $a \neq b$ are constants. Compute the Wroskian $W(f, g)$, and use this to explain why they are linearly independent.

(2) Consider the functions $f(x) = \sin(ax)$ and $g(x) = \cos(bx)$, where $a \neq b$ are constants. Compute their Wroskian, and use this to explain why they are linearly independent.

(3) Consider the functions $f(x) = xe^{cx}$ and $g(x) = xe^{dx}$ where $c \neq d$ are constants. Compute their Wroskian, and use this to explain why they are linearly independent.

(4) Consider the functions $f(x) = x^a e^{cx}$ and $g(x) = x^b e^{dx}$ where either $a \neq b$ or $c \neq d$. Compute their Wroskian, and use this to explain why they are linearly independent.

[1] Pronounced roughly "Vronskian", named after the Polish mathematician Józef Wroński.

[2] See Lesson 63 from the textbook for further details, including the definition for looking at > 2 functions.