MATH 302: WEEK 3 WORKSHEET

Today's worksheet is about using differential equations to solve problems from geometry. Suppose you have a collection of curves in the plane. Another curve is called an *orthogonal trajectory* for the collection if everywhere it passes through a curve in the collection it passes through at a right angle. (The name is because orthogonal is a synonym of perpendicular.)

How can we find orthogonal trajectories using differential equations? If you have two curves $y_1(x)$ and $y_2(x)$ then they meet at a right angle if $y_1'(x) = -1/y_2'(x)$ at a coordinate x where they meet. (This is just the formula for when two lines are perpendicular but applied to tangent lines.) So if we have a description of the family of curves y_2 then we can use this to get a differential equation which describes the orthogonal trajectories y_1.

Let's consider the examples of circles centered at the origin. If you a draw a picture it should be clear that the orthogonal trajectories are the lines through the origins. We can check this using differential equations. The circles are described by the equations $x^2+y^2 = k^2$, for a varying parameter $k > 0$. Implicit differentiation and rearranging gives $y' = -x/y$. So if y_1 is an orthogonal trajectory then $y_1' = y_1/x$. This is a separable differential equation, which we can separate to

$$\frac{\mathrm{d}y_1}{y_1} = \frac{\mathrm{d}x}{x}.$$

Now solve this to get $y_1 = cx$ as a 1-parameter family of solutions. These are precisely the lines through the origin (except the vertical line $x = 0$).

Problem. Follow the steps shown above to compute orthogonal trajectories for the following collections of curves. In each case, you should get a differential equation which describes the derivative of the orthogonal trajectories. Once you have this differential equation, you need to determine which of the methods you've learned can be used to solve the equation. For both problems, it is okay—encouraged, even!—to use a computer integral solver, such as can be found on wolframalapha.com,[1] to solve the integrals you get.

(1) Find the orthogonal trajectories for the hyperbolas $x^2 - y^2 = k^2$, where $k > 0$ is a parameter.
(2) Find the orthogonal trajectories for the curves $xy = x^2 + k$, where k is a parameter.

[1]If you type `integrate f(x)` into the prompt it will give you the indefinite integral of `f(x)`.