MATH 302: WEEK 2 WORKSHEET

Today's worksheet is about the following problem, an example of a kind of problem that can be tackled using differential equations.

The basic idea behind this sort of problem is, you can write down an equation which describes the change Δx of a quantity x over a small period of time Δt, and then by taking a limit as $\Delta t \to 0$ you get a differential equation describing $\mathrm{d}x/\mathrm{d}t$. You can then use techniques for solving differential equations to solve this to get an equation describing x in terms of time t. If the quantity x varies continuously as t changes, this gives a good approximation for the real world quantity.

Problem. *You have a* $15,000$ *gallon pool. You want it to have the salinity of seawater,* 35 *parts per thousand of salt per water, which comes out to* 0.3 *lb of salt for each gallon of water. You miscalculated the size of your pool and put in* 5500 *lb of salt, which is too much. To fix this, you drain the pool at the rate of* 5 *gallons per minute while simultaneously filling it with fresh water at the rate of* 5 *gallons per minute. Meanwhile, you constantly stir the water to keep the salt distribution uniform. Determine how long you need to drain the pool and refill it with fresh water to reach the optimal salt level of* 4500 *lb.*

Let s denote the quantity of salt in the pool (in pounds) at time t (in minutes).

(1) Write an equation in terms of s, t, Δs, and Δt which represents the approximate change Δs in the quantity of salt over the course of a small change Δt in time.
(2) Replace Δs and Δt in this equation with $\mathrm{d}s$ and $\mathrm{d}t$ to get a differential equation. (Formally, we describe $\mathrm{d}s/\mathrm{d}t$ by taking a limit as $\Delta t \to 0$.)
(3) Find the general solution to this separable differential equation.
(4) Find the specific solution which satisfies the initial conditions given in the above problem. Use this to then answer the question about how long you need to drain the pool.
(5) Suppose that instead of starting with 5500 lb of salt you had instead started with 7500 lb of salt. In this case, how long would you need to drain the pool for it to reach 4500 lb of salt?
(6) The problem setup states that you continually stir the pool to keep the salt uniformly distributed throughout the pool. Does your solution to this problem depend on the assumption that the salt distribution is uniform? Explain why or why not.