MATH 302: WEEK 15 WORKSHEET

When we looked in lecture at existence/uniqueness theorems for differential equations, we turned a higher-order differential equation in one function into a system of equations in multiple functions. For this worksheet, we'll look at an application where you want to use multiple equations in multiple functions. Namely, we look at the Lotka–Volterra model from ecology.

The independent variable here is time t and the two dependent variables represent the population of two species, one a predator and the other prey. Let's use x for the prey population and y for the predator population. The model makes the simplifying assumption that the populations only depend on the interaction between the two species: the prey species always has sufficient food available, the predator species's only food source is the prey species, the environment doesn't change over time and there's no evolution in the species.

A number of parameters control the system:

- r, the rate at which the prey population increases in the presence of zero predators, relative to its size;
- g, a proportionality constant for how often the two species meet (and thus how often a predator can consume a prey);
- c, the rate at which the predator population increases, assuming sufficient access to food; and
- m, the die-off rate of the predator population, relative to its size.

Under these simplifying assumptions, the following pair of equations gives a model for population sizes:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx - gxy$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = cgxy - my$$

Rather than ask you to solve these—we haven't learned methods for solving systems of differential equations—for this worksheet I want you to use a visualization tool for these equations. This online tool lets you simulate a Lotka–Volterra system, given initial conditions and values for the parameters.

(1) One way to visualize solutions is to plot both dependent variables relative to the independent variable. (This is the left graph in the tool.) Consider various inputs to the systems to answer the following.
   - What happens after the prey population gets very small?
   - What happens after the predator population gets very small?
   - What happens after the prey population gets very large?
   - What happens after the predator population gets very large?

(2) Another way to visualize solutions is plot it in *phase space*, i.e. to plot pairs $(x(t), y(t))$ for different values of t, seeing what is possible at different lengths of time. In effect, this is plotting a parameterized curve in the 2-dimensional plane. (This is the right graph in the tool.) Considering various inputs to the system, what shapes can you get for the solution plotted in phase space?

(3) Consider some extreme possible inputs to the system. What happens given these inputs? (For example, what if one population starts out at 0? What if one of the parameters are 0?)