MATH 302: WEEK 13 WORKSHEET

This week in class we've been talking about series methods for solving and approximating solutions to differential equations. An important topic we've not covered yet is determining the error in approximations. If you want to approximate a solution, you need to know how accurate your approximation is, or else what good is it?

Fortunately, there are methods to determine how bad the error can be. The basic result is given by *Taylor's theorem*:

Theorem (Taylor)**.** *Suppose a function $f(x)$ is infinitely differentiable on an interval centered at a point p. Then the* remainder term $R_n(x)$ *for the n-th order approximation is bounded by*

$$|R_n(x)| \leq \frac{M(x-p)^{n+1}}{(n+1)!}$$

where M is the maximum value of $|f^{(n+1)}(t)|$ for t in the interval between p and x. In other words, if you compute up to the $(x-p)^n$ term for the power series, the error from the true value is bounded by $\frac{M(x-p)^{n+1}}{(n+1)!}$.

The problem with this error bound is that it's based on having a bound for the $(n+1)$-st derivative. Other, better numerical methods (which we won't learn in this class) avoid this problem.

For this problems, consider the differential equation

$$y'' + xy' + (1+x)y = \cos(x) - 1, \qquad y(0) = 0, y'(0) = 1/2$$

Let's suppose that experimental results suggest that when $x \approx 0$ that the absolute value of y and its derivatives are smaller than 1. (So we can take $M = 1$ as an upper bound for calculating the error.)

(1) Calculate the degree 4 and degree 5 approximations y_4 and y_5 for the power series of the solution.
(2) Compute the approximations $y_4(0.1)$ and $y_5(0.1)$. Give your answers with seven significant digits.
(3) Taking $M = 1$, compute an upper bound for the errors $R_4(0.1)$ and $R_5(0.1)$. Give your answers with seven significant digits.