Math 302: The method of variation of parameters

Kameryn J Williams

University of Hawaiʻi at Mānoa

Spring 2021

Non-homogeneous equations

Just before spring break, we learned one method to solve non-homogeneous linear differential equations with constant coefficients:

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = b(x).$$

That was the method of undetermined coefficients.

Non-homogeneous equations

Just before spring break, we learned one method to solve non-homogeneous linear differential equations with constant coefficients:

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = b(x).$$

That was the method of undetermined coefficients.

This week we will learn a second method to solve these types of equations, the method of variation of parameters.

Non-homogeneous equations

Just before spring break, we learned one method to solve non-homogeneous linear differential equations with constant coefficients:

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = b(x).$$

That was the method of undetermined coefficients.

This week we will learn a second method to solve these types of equations, the method of variation of parameters.

Why learn both?

Non-homogeneous equations

Just before spring break, we learned one method to solve non-homogeneous linear differential equations with constant coefficients:

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = b(x).$$

That was the method of undetermined coefficients.

This week we will learn a second method to solve these types of equations, the method of variation of parameters.

Why learn both? Some equations are more easily solved by one than the other, so it's nice to know both.

Non-homogeneous equations

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = b(x)$$

Non-homogeneous equations

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = b(x)$$

As we've previously discussed, the general solution to this is of the form

$$y_c(x) + y_p(x),$$

where y_c is the general solution to the homogeneous equation

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0$$

and y_p is a particular solution to the non-homogeneous equation.

Non-homogeneous equations

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = b(x)$$

As we've previously discussed, the general solution to this is of the form

$$y_c(x) + y_p(x),$$

where y_c is the general solution to the homogeneous equation

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0$$

and y_p is a particular solution to the non-homogeneous equation.

Like the method of undetermined coefficients, the method of variation of parameters is a method to determine y_p. As before, you have to find y_c by the methods we learned previously. (Find the roots of the corresponding characteristic polynomial, etc.)

An example

$$y'' + y' - 2y = x$$

An example

$$y'' + y' - 2y = x$$

First solve the homogeneous equation.

$$y'' + y' - 2y = 0$$

Get $y_c = c_1 e^x + c_2 e^{-2x}$.

An example

$$y'' + y' - 2y = x$$

First solve the homogeneous equation.

$$y'' + y' - 2y = 0$$

Get $y_c = c_1 e^x + c_2 e^{-2x}$. For y_p, let's guess it takes the form

$$y_p(x) = u(x)e^x + v(x)e^{-2x}.$$

An example

$$y'' + y' - 2y = x$$

First solve the homogeneous equation.

$$y'' + y' - 2y = 0$$

Get $y_c = c_1 e^x + c_2 e^{-2x}$. For y_p, let's guess it takes the form

$$y_p(x) = u(x)e^x + v(x)e^{-2x}.$$

Now figure out the unknown functions u and v.

- $y_p = ue^x + ve^{-2x}$

An example

$$y'' + y' - 2y = x$$

First solve the homogeneous equation.

$$y'' + y' - 2y = 0$$

Get $y_c = c_1 e^x + c_2 e^{-2x}$. For y_p, let's guess it takes the form

$$y_p(x) = u(x)e^x + v(x)e^{-2x}.$$

Now figure out the unknown functions u and v.

- $y_p = ue^x + ve^{-2x}$
- $y_p' = ue^x + v(-2e^{-2x}) + (u'e^x + v'e^{-2x})$.

An example

$$y'' + y' - 2y = x$$

First solve the homogeneous equation.

$$y'' + y' - 2y = 0$$

Get $y_c = c_1 e^x + c_2 e^{-2x}$. For y_p, let's guess it takes the form

$$y_p(x) = u(x)e^x + v(x)e^{-2x}.$$

Now figure out the unknown functions u and v.

- $y_p = ue^x + ve^{-2x}$
- $y_p' = ue^x + v(-2e^{-2x}) + (u'e^x + v'e^{-2x})$.
- $y_p'' = ue^x + v(4e^{-2x}) + (u'e^x + v'(-2e^{-2x})) + (u'e^x + v'e^{-2x})'$.

An example

$$y'' + y' - 2y = x$$

First solve the homogeneous equation.

$$y'' + y' - 2y = 0$$

Get $y_c = c_1 e^x + c_2 e^{-2x}$. For y_p, let's guess it takes the form

$$y_p(x) = u(x)e^x + v(x)e^{-2x}.$$

Now figure out the unknown functions u and v.

- $y_p = ue^x + ve^{-2x}$
- $y_p' = ue^x + v(-2e^{-2x}) + (u'e^x + v'e^{-2x})$.
- $y_p'' = ue^x + v(4e^{-2x}) + (u'e^x + v'(-2e^{-2x})) + (u'e^x + v'e^{-2x})'$.

Now substitute into the equation:

$$\begin{array}{llll} u(e^x) & + v(4e^{-2x}) & + (u'e^x + v'(-2e^{-2x})) & + (u'e^x + v'e^{-2x})' \\ u(e^x) & + v(-2e^{-2x}) & + & (u'e^x + v'e^{-2x}) = x \\ u(-2e^x) & + v(-2e^{-2x}) & & \end{array}$$

The same example, but this takes so long we need a second slide

$$y'' + y' - 2y = x$$

The same example, but this takes so long we need a second slide

$$y'' + y' - 2y = x$$

We got the complementary solution
$y_c = c_1 e^x + c_2 e^{-2x}$
and guessed the particular solution looks like
$y_p = ue^x + ve^{-2x}$.
And then determined we need u and v to satisfy:

$$\begin{aligned} u' e^x + v'(-2e^{-2x}) &= x \\ u' e^x + v'(\quad e^{-2x}) &= 0 \end{aligned}$$

The same example, but this takes so long we need a second slide

$$y'' + y' - 2y = x$$

We got the complementary solution
$y_c = c_1 e^x + c_2 e^{-2x}$
and guessed the particular solution looks like
$y_p = ue^x + ve^{-2x}$.
And then determined we need u and v to satisfy:

$$\begin{aligned} u'e^x + v'(-2e^{-2x}) &= x \\ u'e^x + v'(\quad e^{-2x}) &= 0 \end{aligned}$$

This isn't too bad.

The same example, but this takes so long we need a second slide

$$y'' + y' - 2y = x$$

We got the complementary solution
$y_c = c_1 e^x + c_2 e^{-2x}$
and guessed the particular solution looks like
$y_p = ue^x + ve^{-2x}$.
And then determined we need u and v to satisfy:

$$u'e^x + v'(-2e^{-2x}) = x$$
$$u'e^x + v'(\quad e^{-2x}) = 0$$

This isn't too bad.

$$u' = \frac{1}{3}xe^{-x}$$
$$v' = -\frac{1}{3}xe^{2x}$$

The same example, but this takes so long we need a second slide

$$y'' + y' - 2y = x$$

We got the complementary solution $y_c = c_1 e^x + c_2 e^{-2x}$ and guessed the particular solution looks like $y_p = ue^x + ve^{-2x}$.

And then determined we need u and v to satisfy:

$$\begin{aligned} u'e^x + v'(-2e^{-2x}) &= x \\ u'e^x + v'(\quad e^{-2x}) &= 0 \end{aligned}$$

This isn't too bad.

$$\begin{aligned} u' &= \quad \tfrac{1}{3}xe^{-x} \\ v' &= -\tfrac{1}{3}xe^{2x} \end{aligned}$$

This gives

$$\begin{aligned} u &= e^{-x}\left(-\tfrac{1}{3} - \tfrac{x}{3}\right) \\ v &= e^{2x}\left(\tfrac{1}{12} - \tfrac{x}{6}\right) \end{aligned}$$

The same example, but this takes so long we need a second slide

$$y'' + y' - 2y = x$$

We got the complementary solution
$y_c = c_1 e^x + c_2 e^{-2x}$
and guessed the particular solution looks like
$y_p = ue^x + ve^{-2x}$.
And then determined we need u and v to satisfy:

$$\begin{aligned} u'e^x + v'(-2e^{-2x}) &= x \\ u'e^x + v'(\quad e^{-2x}) &= 0 \end{aligned}$$

This isn't too bad.

$$\begin{aligned} u' &= \frac{1}{3}xe^{-x} \\ v' &= -\frac{1}{3}xe^{2x} \end{aligned}$$

This gives

$$\begin{aligned} u &= e^{-x}\left(-\frac{1}{3} - \frac{x}{3}\right) \\ v &= e^{2x}\left(\frac{1}{12} - \frac{x}{6}\right) \end{aligned}$$

Plugging in and simplifying gives $y_p = -\frac{x}{2} - \frac{1}{4}$.

Why do all that work when we already knew how solve that equation?

Why do all that work when we already knew how solve that equation?

- With the method of undetermined coefficients, the idea was to guess that the particular solution looks like a linear combination of terms from the right-hand side and their derivatives.
- So this only works if these terms only have finitely many linearly independent derivatives.

Why do all that work when we already knew how solve that equation?

- With the method of undetermined coefficients, the idea was to guess that the particular solution looks like a linear combination of terms from the right-hand side and their derivatives.
- So this only works if these terms only have finitely many linearly independent derivatives.
- In the example, we never used that the right-hand side x only has finitely many linearly independent derivatives.
- So this method generalizes to more settings than the other method.

Why do all that work when we already knew how solve that equation?

- With the method of undetermined coefficients, the idea was to guess that the particular solution looks like a linear combination of terms from the right-hand side and their derivatives.
- So this only works if these terms only have finitely many linearly independent derivatives.
- In the example, we never used that the right-hand side x only has finitely many linearly independent derivatives.
- So this method generalizes to more settings than the other method.

So let's talk about how to do this method in general. This is the same process as we did with the example, we just have to state it in more general terms.

We'll first do a step by step walk through the general process, then summarize the formulae you get at the end.

The method of variation of parameters

We want to solve the equation

$$a_2 y'' + a_1 y' + a_0 y = b(x).$$

The method of variation of parameters

We want to solve the equation

$$a_2 y'' + a_1 y' + a_0 y = b(x).$$

Start by determining the complementary solution

$$y_c = c_1 y_1 + c_2 y_2.$$

The method of variation of parameters

We want to solve the equation

$$a_2 y'' + a_1 y' + a_0 y = b(x).$$

Start by determining the complementary solution

$$y_c = c_1 y_1 + c_2 y_2.$$

Now guess that a particular solution is

$$y_p = u y_1 + v y_2.$$

We need to solve for u and v.

To do this, we calculate y_p' and y_p'' and plug into the equation.

The method of variation of parameters

We want to solve the equation

$$a_2 y'' + a_1 y' + a_0 y = b(x).$$

Start by determining the complementary solution

$$y_c = c_1 y_1 + c_2 y_2.$$

Now guess that a particular solution is

$$y_p = u y_1 + v y_2.$$

We need to solve for u and v.

To do this, we calculate y_p' and y_p'' and plug into the equation.

- $y_p = u y_1 + v y_2$
- $y_p' = u y_1' + v y_2' + (u' y_1 + v' y_2)$
- $y_p'' = u y_1'' + v y_2'' + (u' y_1' + v' y_2') + (u' y_1 + v' y_2)'$

The method of variation of parameters

We want to solve the equation

$$a_2 y'' + a_1 y' + a_0 y = b(x).$$

Start by determining the complementary solution

$$y_c = c_1 y_1 + c_2 y_2.$$

Now guess that a particular solution is

$$y_p = u y_1 + v y_2.$$

We need to solve for u and v.

To do this, we calculate y_p' and y_p'' and plug into the equation.

- $y_p = u y_1 + v y_2$
- $y_p' = u y_1' + v y_2' + (u' y_1 + v' y_2)$
- $y_p'' = u y_1'' + v y_2'' + (u' y_1' + v' y_2') + (u' y_1 + v' y_2)'$

We're gonna need a new slide for the space to write the substitution.

The method of variation of parameters, continued

Substitute $y_p = uy_1 + vy_2$ into $a_2y'' + a_1y' + a_0y = b(x)$:

$$\begin{aligned} &u(a_2y_1'') + v(a_2y_2'') + a_2(u'y_1' + v'y_2') + a_2(u'y_1 + v'y_2)' \\ &u(a_1y_1') + v(a_1y_2') + \qquad\qquad a_1(u'y_1 + v'y_2) = b(x) \\ &u(a_0y_1) + v(a_0y_2) \end{aligned}$$

The method of variation of parameters, continued

Substitute $y_p = uy_1 + vy_2$ into $a_2y'' + a_1y' + a_0y = b(x)$:

$$\begin{array}{llll} u(a_2y_1'') & + v(a_2y_2'') & + a_2(u'y_1' + v'y_2') & + a_2(u'y_1 + v'y_2)' \\ u(a_1y_1') & + v(a_1y_2') & + & a_1(u'y_1 + v'y_2) = b(x) \\ u(a_0y_1) & + v(a_0y_2) & & \end{array}$$

This simplifies down to

$$\begin{aligned} a_2(u'y_1' + v'y_2') + & a_2(u'y_1 + v'y_2)' \\ & a_1(u'y_1 + v'y_2) = b(x) \end{aligned}$$

The method of variation of parameters, continued

Substitute $y_p = uy_1 + vy_2$ into $a_2y'' + a_1y' + a_0y = b(x)$:

$$\begin{aligned} u(a_2y_1'') &+ v(a_2y_2'') &+ a_2(u'y_1' + v'y_2') &+ a_2(u'y_1 + v'y_2)' \\ u(a_1y_1') &+ v(a_1y_2') &+ &\quad a_1(u'y_1 + v'y_2) = b(x) \\ u(a_0y_1) &+ v(a_0y_2) \end{aligned}$$

This simplifies down to

$$\begin{aligned} a_2(u'y_1' + v'y_2') + &a_2(u'y_1 + v'y_2)' \\ &a_1(u'y_1 + v'y_2) = b(x) \end{aligned}$$

This equation will be true when
$u'y_1' + v'y_2' = b(x)/a_2$ and
$u'y_1 + v'y_2 = 0$.

The method of variation of parameters, continued

Substitute $y_p = uy_1 + vy_2$ into $a_2 y'' + a_1 y' + a_0 y = b(x)$:

$$\begin{aligned} &u(a_2 y_1'') + v(a_2 y_2'') + a_2(u'y_1' + v'y_2') + a_2(u'y_1 + v'y_2)' \\ &u(a_1 y_1') + v(a_1 y_2') + \qquad a_1(u'y_1 + v'y_2) = b(x) \\ &u(a_0 y_1) + v(a_0 y_2) \end{aligned}$$

This simplifies down to

$$\begin{aligned} a_2(u'y_1' + v'y_2') + a_2(u'y_1 + v'y_2)' & \\ a_1(u'y_1 + v'y_2) &= b(x) \end{aligned}$$

This equation will be true when
$u'y_1' + v'y_2' = b(x)/a_2$ and
$u'y_1 + v'y_2 = 0.$

You can solve this pair of equations for u and v by hand each time.

But using tools from linear algebra we can describe a general formulae that always works.

The method of variation of parameters, continued

Substitute $y_p = uy_1 + vy_2$ into $a_2y'' + a_1y' + a_0y = b(x)$:

$$\begin{aligned} u(a_2y_1'') &+ v(a_2y_2'') &+ a_2(u'y_1' + v'y_2') &+ a_2(u'y_1 + v'y_2)' \\ u(a_1y_1') &+ v(a_1y_2') &+ & a_1(u'y_1 + v'y_2) = b(x) \\ u(a_0y_1) &+ v(a_0y_2) \end{aligned}$$

This simplifies down to

$$a_2(u'y_1' + v'y_2') + a_2(u'y_1 + v'y_2)' + a_1(u'y_1 + v'y_2) = b(x)$$

This equation will be true when
$u'y_1' + v'y_2' = b(x)/a_2$ and
$u'y_1 + v'y_2 = 0$.

You can solve this pair of equations for u and v by hand each time.

But using tools from linear algebra we can describe a general formulae that always works.

Recall the formula for the determinant:

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = AD - BC.$$

The method of variation of parameters, continued more

We need to solve the pair of equations

$$u'y_1' + v'y_2' = b(x)/a_2$$
$$u'y_1 + v'y_2 = 0.$$

The method of variation of parameters, continued more

We need to solve the pair of equations

$$u'y_1' + v'y_2' = b(x)/a_2$$
$$u'y_1 + v'y_2 = 0.$$

The general solution is:

$$u' = \frac{\begin{vmatrix} 0 & y_2 \\ b(x)/a_2 & y_2' \end{vmatrix}}{W(y_1, y_2)}$$

and

$$v' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & b(x)/a_2 \end{vmatrix}}{W(y_1, y_2)}$$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

is the Wroskian of y_1 and y_2.

The method of variation of parameters, continued more

We need to solve the pair of equations

$$u'y_1' + v'y_2' = b(x)/a_2$$
$$u'y_1 + v'y_2 = 0.$$

The general solution is:

$$u' = \frac{\begin{vmatrix} 0 & y_2 \\ b(x)/a_2 & y_2' \end{vmatrix}}{W(y_1, y_2)}$$

and

$$v' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & b(x)/a_2 \end{vmatrix}}{W(y_1, y_2)}$$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

is the Wroskian of y_1 and y_2.
(We earlier used Wroskians for checking linear independence. It turns out that since y_1 and y_2 are linearly independent solutions to the same homogeneous ODE, their Wroskian is nonzero, so we aren't dividing by zero.)

The method of variation of parameters, continued more

We need to solve the pair of equations

$$u'y_1' + v'y_2' = b(x)/a_2$$
$$u'y_1 + v'y_2 = 0.$$

The general solution is:

$$u' = \frac{\begin{vmatrix} 0 & y_2 \\ b(x)/a_2 & y_2' \end{vmatrix}}{W(y_1, y_2)}$$

and

$$v' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & b(x)/a_2 \end{vmatrix}}{W(y_1, y_2)}$$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

is the Wroskian of y_1 and y_2.
(We earlier used Wroskians for checking linear independence. It turns out that since y_1 and y_2 are linearly independent solutions to the same homogeneous ODE, their Wroskian is nonzero, so we aren't dividing by zero.)

Now that we know u' and v' we get u and v by antidifferentiating.

The method of variation of parameters, continued more

We need to solve the pair of equations

$$u'y_1' + v'y_2' = b(x)/a_2$$
$$u'y_1 + v'y_2 = 0.$$

The general solution is:

$$u' = \frac{\begin{vmatrix} 0 & y_2 \\ b(x)/a_2 & y_2' \end{vmatrix}}{W(y_1, y_2)}$$

and

$$v' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & b(x)/a_2 \end{vmatrix}}{W(y_1, y_2)}$$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

is the Wroskian of y_1 and y_2.
(We earlier used Wroskians for checking linear independence. It turns out that since y_1 and y_2 are linearly independent solutions to the same homogeneous ODE, their Wroskian is nonzero, so we aren't dividing by zero.)

Now that we know u' and v' we get u and v by antidifferentiating.

We then substitute these in to find the particular solution

$$y_p = uy_1 + vy_2.$$

That was a lot of steps, so let's summarize

We want to solve the equation

$$a_2 y'' + a_1 y' + a_0 y = b(x),$$

where $b(x)$ is any function of x.

That was a lot of steps, so let's summarize

We want to solve the equation

$$a_2 y'' + a_1 y' + a_0 y = b(x),$$

where $b(x)$ is any function of x.

- First, find the complementary solution $y_c = c_1 y_1 + c_2 y_2$.
- Then, guess that a particular solution looks like $y_p = u y_1 + v y_2$.

That was a lot of steps, so let's summarize

We want to solve the equation

$$a_2 y'' + a_1 y' + a_0 y = b(x),$$

where $b(x)$ is any function of x.

- First, find the complementary solution $y_c = c_1 y_1 + c_2 y_2$.
- Then, guess that a particular solution looks like $y_p = u y_1 + v y_2$.
- We determined formulae to calculate u' and v'.

$$u' = \frac{-y_2 b(x)/a_2}{W(y_1, y_2)}, \qquad v' = \frac{y_1 b(x)/a_2}{W(y_1, y_2)}$$

where the denominators are the Wroskian

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

That was a lot of steps, so let's summarize

We want to solve the equation

$$a_2 y'' + a_1 y' + a_0 y = b(x),$$

where $b(x)$ is any function of x.

- First, find the complementary solution $y_c = c_1 y_1 + c_2 y_2$.
- Then, guess that a particular solution looks like $y_p = u y_1 + v y_2$.
- We determined formulae to calculate u' and v'.

$$u' = \frac{-y_2 b(x)/a_2}{W(y_1, y_2)}, \qquad v' = \frac{y_1 b(x)/a_2}{W(y_1, y_2)}$$

where the denominators are the Wroskian

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

- Antidifferentiate to calculate u and v.

That was a lot of steps, so let's summarize

We want to solve the equation

$$a_2 y'' + a_1 y' + a_0 y = b(x),$$

where $b(x)$ is any function of x.

- First, find the complementary solution $y_c = c_1 y_1 + c_2 y_2$.
- Then, guess that a particular solution looks like $y_p = u y_1 + v y_2$.
- We determined formulae to calculate u' and v'.

$$u' = \frac{-y_2 b(x)/a_2}{W(y_1, y_2)}, \qquad v' = \frac{y_1 b(x)/a_2}{W(y_1, y_2)}$$

where the denominators are the Wroskian

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

- Antidifferentiate to calculate u and v.
- Substitute your solutions for u, v, y_1, and y_2 to get the particular solution

$$y_p = u y_1 + v y_2.$$

An example

$$y'' + y = \tan x, \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

An example

$$y'' + y = \tan x, \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$y_c = c_1 \cos x + c_2 \sin x$

Another example

$$y'' + 2y + y = e^{-x} \log x$$

Another example

$$y'' + 2y + y = e^{-x}\log x$$

$y_c = c_1 e^{-x} + c_2 x e^{-x}$

What about ODEs of order > 2?

- The method we learned only applies to second-order ODEs.
- Can we generalize it to higher order equations?

What about ODEs of order > 2?

- The method we learned only applies to second-order ODEs.
- Can we generalize it to higher order equations?

Yes we can!

What about ODEs of order > 2?

- The method we learned only applies to second-order ODEs.
- Can we generalize it to higher order equations?

Yes we can!

- It is, however, more complicated.
- Instead of having two unknown functions u and v to solve for, if your equation is order n you have n unknown functions to solve for.
- Similar to the 2 case, you have n many equations in these unknown functions, with the known part of the equations based on the n parts of y_c and their derivatives, up to order $n-1$.

What about ODEs of order > 2?

- The method we learned only applies to second-order ODEs.
- Can we generalize it to higher order equations?

Yes we can!

- It is, however, more complicated.
- Instead of having two unknown functions u and v to solve for, if your equation is order n you have n unknown functions to solve for.
- Similar to the 2 case, you have n many equations in these unknown functions, with the known part of the equations based on the n parts of y_c and their derivatives, up to order $n - 1$.

But we won't cover the more general method in this class. If you're interested, page 235 of the textbook has the equations you would need to solve.