Math 302: The method of undetermined coefficients

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Non-homogeneous equations

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Starting this week, we will talk about how to solve nonhomogenous equations.

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where y_c is the general solution to the homogeneous equation

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This week is about one method for finding y_p, the method of undetermined coefficients. It only applies to the case where $b(x)$ is a combination of terms of the form

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Specifically, what we need is that $b(x)$ is a sum of terms each of which has only a finite number of linearly independent derivatives.

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So getting $x^2 - x$ on the righthand side cannot come from the solution to the homogeneous equation.

Another example

$$y'' + 4y' = e^{-x}$$

Yet another example

$$y'' - 2y' + 2y = \sin(2x) + 2\cos(x)$$

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- If you have a product, you need to look at all its linearly independent derivatives. For example, if x^2e^{ax} is part of $b(x)$, then $Ax^2e^{ax} + Bxe^{ax} + Ce^{ax}$ is part of y_p.

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In all cases, you need to determine the exact value of the coefficients $A, B, C, \ldots$.

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$y_c = c_1 e^x + c_2 e^{-x}$. It cannot be that $y_p = Ae^x$, as that would be a solution to $y'' - y = 0$. Instead, y_p depends on xe^x, similar to the repeated roots case for homogeneous equations.

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$y_c = c_1 \cos x + c_2 \sin x$. y_p will have an $x \cos x$ term. Its derivative is $\cos x + x \sin x$, so we also need an $x \sin x$ term.

The general case

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = b(x)$$

- Suppose the characteristic polynomial has a root r with multiplicity m, possibly $m > 1$.
- Let $t(x)$ be the term of y_c coming from this root r. (So $t(x)$ is an exponential function, trig function, or product of the two.)
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And you have to do this for each time this happens.

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- So then it has to contain a $1/x^2$ term.
- So it also has to contain a $1/x^3$ term.
- Continuing this pattern y_p has to contain a $1/x^n$ term for every positive integer n.

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- This is a homogeneous linear differential equation with constant coefficients. We know what solutions to these look like.
- They look like combinations of $a, x^k, e^{ax}, \cos(ax), \sin(ax)$.
- So that's why this method only works when $b(x)$ is a combination of these.