Math 302: Series methods

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Undetermined coefficients

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$$y'' + y' + y = x^3$$

we guessed that a particular solution looks like

$$A + Bx + Cx^2 + Dx^3$$

Then we solve for the values of the coefficients.

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That is, we want to see what we can figure out if we represent the solution as a power series.

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A power series centered at p has a radius of convergence R:

- If $R = 0$ the series converges iff $x = p$.
- If $0 < R < \infty$ the series converges if $|x - p| < R$. At the end points $x = p \pm R$ it may either converge or diverge
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You can use convergence tests like you learned in Calc II to figure out the radius of convergence of a given power series.

From power series to functions

$$\sum_{n=0}^{\infty} a_n x^n$$

If the interval of convergence of this power series is nontrivial (i.e. $R > 0$), then the power series defines a continuous function on the interval $p - R < x < p + R$:

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$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{k=0}^{\infty} (k+1) x_{k+1} x^k$$

That is, you get the derivative by differentiating the power series term by term.

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This works centered at p, not just centered at 0.

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So we can write:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

for the power series for $f(x)$. We call this its Taylor series.

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Let's remember the Taylor series centered at 0 for some important functions:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$
$$\cos x = 0 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \pm \cdots$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \pm \cdots$$

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Let's plug these into the equation:

$$\begin{aligned} &x(a_1 + 2a_2x + 3a_3x^2 + \cdots) \\ -&(a_0 + \ \ a_1x + \ \ a_2x^2 + \cdots) = 0 + 0x + 0x^2 + \cdots \end{aligned}$$

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So the solution is $y = a_1x$, where a_1 is an arbitrary constant.

An existence theorem

$$y^{(n)} + \cdots + a_1(x)y' + a_0(x) = b(x)$$

If the coefficient functions $a_i(x)$ and the function $b(x)$ are all analytic on the same interval centered on p, then there is a unique solution satisfying the initial conditions

$$y(p) = v_0, \quad y'(p) = v_1, \quad \cdots, \quad y^{(n-1)}(p) = v_{n-1}$$

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In short, this result tells us that if all the parts of the equation are analytic functions, then it's valid to guess that the solution is given by a power series and use that to determine the solution.

In particular, if the functions are all polynomials, exponential functions, sine/cosine, or combinations thereof, then we get a solution which is valid for all of $\mathbb{R}$.

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Solving for the first few terms:

$$y = 1 + x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24} + \cdots$$

Another example

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where a_0 and a_1 are arbitrary constants. This doesn't give us a nice way to write the solution in terms of elementary functions, but why should we expect to always be able to do so?

A more complicated example, finding an approximation

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For this one, we have to multiply two power series!

Singular versus ordinary points

$$y'' + a(x)y' + b(x)y = c(x)$$

- Write your equation in the standard form where the leading coefficient is 1.
- Look at the coefficient functions $a(x)$, $b(x)$, and $c(x)$. If they are all analytic at a point p then p is an ordinary point.
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For our purposes, it's important to recognize singular points so you know where the methods we do learn don't apply.