Math 302: Operators

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Previously

We've been learning various methods for solving linear differential equations.

The next big topic is the Laplace transform which gives, among other things, yet another method for solving these.

But before we talk about the Laplace transform, let's step back a bit to take a more general view on operators.

This isn't new content per se, but rather is placing content we've already learned into a more general framework.

Operators

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Another important example is integration:

$$\int_{x_0}^{x} f(t)\,\mathrm{d}t$$

converts a function f into a new function.

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- $S + T$, where S and T are linear
- aT, where T is linear and a is a constant

Polynomial operators

Consider a polynomial

$$P(r) = a_n r^n + \cdots + a_1 r + a_0.$$

Then if T is a linear operator, so is

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If you've taken linear algebra: this is like when you find kernels of matrices, but in an infinite dimensional vector space.

The principle of superposition

Recall that the general solution to

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Suppose T is a linear operator and consider the equation

$$Ty = a_1 b_1 + \cdots + a_n b_n,$$

where a_i are constants and b_i are functions. Suppose you have solutions y_i to the equations

$$Ty = b_i.$$

Then, $a_1 y_1 + \cdots + a_n y_n$ is a solution to the original equation.

In other words: you can get the solution by breaking up the sum into pieces and solving each piece individually. For example, this is how you differentiate.

The algebra of linear operators

Linear operators satisfy most of the algebraic properties of the integers, where addition and multiplication of operators are defined as:

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The exception is commutativity of multiplication: in general, ST and TS won't be the same operator. However, if they are polynomial operators $P(D)$ and $Q(D)$, then they will be the same:

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The distributive law, associativity of addition and multiplication, and commutativity of addition all hold:

$$\begin{aligned} S(T+U) &= ST + SU \\ (S+T)U &= SU + TU \\ (S+T)+U &= S+(T+U) \\ (ST)U &= S(TU) \\ S+T &= T+S \end{aligned}$$

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$$(ST)U = S(TU)$$
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(In math jargon: linear operators form a noncommutative ring, and restricting to the polynomial operators gives a commutative ring.)

Exponential shift

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To see why this is true, it's enough to check that $D(ue^{ax}) = e^{ax}(D + a)u$. Once we see that, the similar fact for D^n is true by repeatedly applying it, and then it holds for any polynomial operator by taking a linear combination.

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This can be seen by a calculation using the product rule:

$$D(ue^{ax}) = \frac{\mathrm{d}}{\mathrm{d}x}ue^{ax}$$

Solving linear ODEs with operators

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- Consider the ODE $a_n y^{(n)} + \cdots + a_1 y' + a_0 y = b(x)$.
- This is in the form $P(D)y = b(x)$, where $P(r) = a_n r^n + \cdots a_1 r + a_0$.
- You can solve $P(D)y = b(x)$ by determing an inverse operator for $P(D)$.
- Compare to how you solve $Dy = b(x)$ by means of the inverse operator for differentiation, namely integration.