Math 302: Differential equations with linear coefficients

Kameryn J Williams

University of Hawaiʻi at Mānoa

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A differential equation

$$(x+y)\,\mathrm{d}x + (2x-y)\,\mathrm{d}y = 0$$

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$$(x + y)\,dx + (2x - y)\,dy = 0$$

As was mentioned on Monday, this is an instance of an ODE with homogeneous coefficients. So let's solve it.

A more general equation

$$(ax + by)\,\mathrm{d}x + (Ax + By)\,\mathrm{d}y = 0$$

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$$\frac{\mathrm{d}x}{x} + \frac{A + Bu}{a + (b-1)u}\,\mathrm{d}u = 0$$

What if there are constant terms?

$$(x + y + 1)\,\mathrm{d}x + (x - y - 1)\,\mathrm{d}y = 0$$

Some geometry of lines

$$x + y + 1 = 0 \qquad\qquad x - y - 1 = 0$$

What if there are constant terms?

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A general algorithm

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1. Find the intersection $(\bar{x}, \bar{y})$ of the two lines, then transform to

$$(a\bar{x} + b\bar{y})\,\mathrm{d}\bar{x} + (A\bar{x} + B\bar{y})\,\mathrm{d}\bar{y} = 0$$

2. Then use the substitution $u = \bar{y}/\bar{x}$ to transform it into a separable equation in $\bar{x}$ and u:

$$\frac{\mathrm{d}\bar{x}}{\bar{x}} + \frac{A + Bu}{a + (b-1)u}\,\mathrm{d}u = 0$$

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3. You can then solve by integrating both sides. For the u part, you will have do integration by partial fractions like you learned in calc II.

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4. When you have a solution in terms of $\bar{x}$ and u, back substitute to get an answer in terms of the original x and y.

What about parallel lines?

If the two lines $ax + by + c = 0$ and $Ax + By + C = 0$ are parallel, they don't have a point of intersection, so the previous method won't work.

An example

$$(2x + 3y - 1)\,\mathrm{d}x + (4x + 6y + 2)\,\mathrm{d}y = 0$$