Math 302: Linear higher-order ODES

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Higher-order linear differential equations

A higher-order differential equation is linear if it can be put into the following form:

$$a_n(x)y^{(n)} + \cdots + a_1(x)y' + a_0(x)y = b(x).$$

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If we want to solve equations like this one, the first question to ask is whether it's even the case that they must be solvable.

Existence and uniqueness of solutions

Theorem

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Consider this differential equation, where the coefficient functions $a_i(x)$ *and* $b(x)$ *are all continuous on a common interval and the leading coefficient function* $a_n(x)$ *is not identically* 0.

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Then, this equation has a unique solution satisfying the initial conditions

$$y(x_0) = v_0, \quad y'(x_1) = v_1, \quad \cdots \quad y^{(n-1)}(x_0) = v_{n-1},$$

where x_0 *is a point in the interval and* $v_0, v_1, \ldots, v_{n-1}$ *are constants.*

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- For now, we will take this theorem as given. We will talk a bit about its proof at the end of the semester.

Homogeneous linear ODEs

To solve these, we will first focus on those where the function on the right is 0, i.e. the equation is of the form

$$a_n(x)y^{(n)} + \cdots + a_1(x)y' + a_0(x)y = 0.$$

We call these homogeneous linear differential equations.

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1. This equation has n many linearly independent solutions $y_1(x), y_2(x), \ldots, y_n(x)$.

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1. This equation has n many linearly independent solutions $y_1(x), y_2(x), \ldots, y_n(x)$.
2. The linear combination

$$y_c(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$$

is an n-parameter family of solutions for the equation. Indeed, every solution takes this form.

Getting from individual solutions to a general solution

Let's see why knowing n linearly independent solutions gives us an n-parameter family of solutions. To make things readable, let's look at the second-order case.

$$a_2(x)y'' + a_1(x)y' + a_0(x) = 0$$

has solutions $y_1(x)$ and $y_2(x)$.

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- By definition of what it means to be a solution, $a_2y_1'' + a_1y_1' + a_0y_1 = 0$ and $a_2y_2'' + a_1y_2' + a_0y_2 = 0$.

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- Let c_1 and c_2 be constants, then add these equations together:

$$\begin{aligned} c_1(a_2y_1'' + a_1y_1' + a_0y_1) + c_2(a_2y_2'' + a_1y_2' + a_0y_2) &= 0 \\ a_2(c_1y_1'' + c_2y_2'') + a_1(c_1y_1' + c_2y_2'') + a_0(c_1y_1 + c_2y_2) &= 0 \\ a_2y_c'' + a_1y_c' + a_0y_c &= 0 \end{aligned}$$

- So $y_c = c_1y_1 + c_2y_2$ is a solution.

Non-homogeneous differential equations

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If $y_p(x)$ is a particular solution to this equation and $y_c(x)$ is the n-parameter family of solutions to the corresponding homogeneous differential equation

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Let's see why this is.

Families of solutions for non-homogeneous ODEs

Let's see the second-order case.

Suppose that y_p is a solution to

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x)$$

and y_c is a solution to

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0.$$

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$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0.$$

Then, adding two equations together, we get

$$\begin{aligned} a_2y_p'' + a_1y_p' + a_0y_p + a_2y_c'' + a_1y_c' + a_0y_c &= b(x) + 0 \\ a_2(y_p'' + y_c'') + a_1(y_p' + y_c') + a_0(y_p + y_c) &= b(x) \\ a_2Y'' + a_1Y' + a_0Y &= b(x), \end{aligned}$$

where $Y(x) = y_p(x) + y_c(x)$ is this n-parameter family of solutions.

A summary

- To find the general solution to the homogeneous equation

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  we need to find the general solution to the corresponding homogeneous equation, and find one particular solution to the non-homogeneous equation.

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So the main part of the work is in finding the individual solutions.

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We start with homogeneous equations.

An example

Consider the equation

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Any ideas of functions which are linearly dependent with their derivatives?

Another example

$$y'' + 5y' + 6y = 0$$

Again, we need a certain weighted sum of y and its derivatives to cancel out to 0.

A more general example

$$ay'' + by' + cy = 0$$

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Consider a possible solution of the form $y = e^{rx}$. Then,

$$\begin{aligned} y &= e^{rx} \\ y' &= re^{rx} \\ y'' &= r^2 e^{rx} \end{aligned}$$

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Consider a possible solution of the form $y = e^{rx}$. Then,

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Plug it into the equation and combine like terms to get:

$$(ar^2 + br + cr)e^{rx} = 0.$$

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This equation is true if and only if $ar^2 + br + c = 0$. That is, the coefficient r in the exponent needs to be a root of the polynomial $av^2 + bv + c = 0$.

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What's going on???

Complex functions

We saw that $y'' + y = 0$ has the following four solutions:

$$y = e^{ix}$$
$$y = e^{-ix}$$
$$y = \sin x$$
$$y = \cos x$$

But it should only have two linearly independent solutions.

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This is why we want to know the complex connection between circular trig functions and the exponential function:

$$e^{ix} = \cos x + i \sin x$$
$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$
$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

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It turns out that the other solution is $y = xe^{-x}$.

To sum up

To solve the equation

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0$$

we want to find all roots of the characteristic polynomial

$$a_n v^n + \cdots + a_1 v + a_0.$$

If r is a root, then $y = e^{rx}$ is a solution.

If all roots are real and distinct, then we are done! We have n many linearly independent solutions, from which we get the general solution!

But there may be a complication.

1. If there are complex roots, we may want to write their solutions with sin and cos instead of the the complex exponential.
2. There may be repeated roots.

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Let's talk about how to handle these!

Complex roots

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0$$
$$a_n v^n + \cdots + a_1 v + a_0$$

- If $a + bi$ is a complex root of a polynomial with real coefficients, then so is $a - bi$.
- Written in exponential form, this gives $y = e^{(a+bi)x}$ and $y = e^{(a-bi)x}$ as two solutions.
- We can rewrite these as $y = e^a(\cos(bx) + i\sin(bx))$ and $y = e^a(\cos(bx) - i\sin(bx))$.
- Both of these are linear combinations of the solutions $y = e^a\cos(bx)$ and $y = e^a\sin(bx)$.
- So we could instead take these to be the two solutions.
- This is especially nice if we only care about real inputs/outputs, not what's happening on the whole complex plane.

Repeated roots

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0$$
$$a_n v^n + \cdots + a_1 v + a_0$$

- Suppose this polynomial has a root r with multiplicity m.
- Then you get m many solutions from this root:

$$y = e^{rx}, \quad y = xe^{rx}, \quad \ldots \quad y = x^{m-1}e^{rx}$$

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- It could be that $r = a + bi$ is complex, in which case you have to combine the work for both complications. You would also have $a - bi$ as a root with multiplicity m, giving $2m$ many solutions:

$$y = e^{ax}\cos(bx) \quad y = xe^{ax}\cos(bx) \quad \ldots \quad y = x^{m-1}e^{ax}\cos(bx)$$
$$y = e^{ax}\sin(bx) \quad y = xe^{ax}\sin(bx) \quad \ldots \quad y = x^{m-1}e^{ax}\sin(bx)$$