Math 302: The Legendre Differential Equation

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Differential equations in physics

- Many higher order differential equations which get applied are linear equations with constant coefficients.
- For example, harmonic motion is described by the differential equation

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- But not all.
- Let's talk about the Legendre equation

$$(1 - x^2)y'' - 2xy' + k(k+1)y = 0 \qquad (k \in \mathbb{R})$$

which has many applications in physics and engineering.

A first look at the Legendre equation

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$$-\frac{2x}{1-x^2} = -2x(1 + x^2 + x^4 + \cdots)$$

$$\frac{k(k+1)}{1-x^2} = k(k+1)(1 + x^2 + x^4 + \cdots)$$

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So let's try to find a solution centered at 0, valid for the interval $(-1, 1)$.

Solving Legendre

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Guess the solution is given by a power series:

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

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Put these together, to get the coefficients are:

- (constant)

$$2a_2 + k(k+1)a_0 = 0$$

- (x term)

$$6a_3 + (-2 + k(k+1))a_1 = 0$$

- (x^n term)

$$(n+1)(n+2)a_{n+2} + (-n(n+1) + k(k+1))a_n = 0$$

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Recurrence relations for the terms:

$$\begin{aligned} 2a_2 + k(k+1)a_0 &= 0 \\ 6a_3 + (-2 + k(k+1))a_1 &= 0 \\ (n+1)(n+2)a_{n+2} + (-n(n+1) + k(k+1))a_n &= 0 \end{aligned}$$

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This is a start, but we can solve for even terms just in terms of a_0 and odd terms just in terms of a_1.

$$\begin{aligned} a_2 &= -\frac{k(k+1)}{2}a0 \\ a_4 &= \frac{6 - k(k+1)}{3 \cdot 4}a_2 \\ &= -\frac{(k-2)(k+3)}{3 \cdot 4}a_2 \\ &= \frac{k(k+1)(k-2)(k+3)}{4!}a_0 \end{aligned}$$

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$$a_2 = -\frac{k(k+1)}{2}a0$$
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In general:

$$a_{2n} = (-1)^n \frac{k(k+1)(k-2)(k+3)(k-4)(k+5)\cdots(k+2n-1)}{(2n)!}a_0$$

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A similar idea gives the odd-indexed coefficients, but I'll skip the details:

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$$a_{2n+1} = \underbrace{(-1)^n \frac{(k-1)(k+2)(k-3)(k+4)(k-5)(k+6)\cdots(k+2n)}{(2n+1)!}}_{=O_{2n+1}} a_1$$

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So the general solution, valid in the interval $(-1, 1)$, is:

$$y = a_0\mathcal{E}(x) + a_1\mathcal{O}(x) = a_0 \sum_{n \text{ even}} E_n x^n + a_1 \sum_{n \text{ odd}} O_n x^n$$

The role of k

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If k is not an integer, both solutions are given by power series with infinitely many terms.

Legendre polynomials

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For integers $k \geq 0$ we call the polynomial solutions $a_0\mathcal{E}(x)$ or $a_1\mathcal{O}(x)$ to the Legendre equation the Legendre polynomials.

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Here's the first few Legendre polynomials.

$$p_0(x) = 1$$
$$p_1(x) = x$$
$$p_2(x) = \frac{1}{2}(3x^2 - 1)$$
$$p_3(x) = \frac{1}{2}(5x^3 - 3x)$$
$$p_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$
$$p_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

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- Thus, the Legendre polynomials form an orthogonal basis for the polynomial functions on $[-1, 1]$.
- More, any differentible function on $[-1, 1]$ can be approximated with weighted sums of Legendre polynomials.