Math 302: The Laplace Transform, II

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Let's remember the definition

$f(t)$ is a function in the time domain t $(0 \leq t < \infty)$. Its Laplace transform $\mathcal{L}[f]$ is a function in the frequency domain s defined as:

$$\mathcal{L}[f](s) = \int_0^\infty f(t)e^{-st}\,\mathrm{d}t.$$

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So this simplifies to

$$\mathcal{L}[y'] = s\mathcal{L}[y] - y(0).$$

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Since the Laplace transform is a linear operator, we can lift this calculation to polynomial operator applied to y:

- Suppose $P(D)$ is a polynomial operator. Then,
- $\mathcal{L}[P(D)y] = P(s)\mathcal{L}[y] - Q(s)$,
- where $Q(s)$ is a polynomial in s depending on the coefficients of the polynomial and the values of y and its derivatives at 0.

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In a slogan: the Laplace transform turns differentiation into multiplication, turning a differential equation into a polynomial equation.

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$$\mathcal{L}[y] = \frac{4}{s - \frac{2}{3}}.$$

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Consult a table of Laplace transforms to get

$$y = 4e^{2t/3}$$

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Now apply the inverse Laplace transform, using a table for each piece

$$y = \frac{3}{10}\cos x + \frac{1}{10}\sin x + \frac{17}{10}e^{-3t}.$$

The general template

$$P(D)y = a_n y^{(n)} + \cdots + a_1 y' + a_0 y = b(t), \qquad y(0) = \cdots$$

1. Take the Laplace transform of both sides, giving you:

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3. Use partial fraction decomposition to rewrite the right-hand side as as sum of simple fractions. (Computers are great for this!)
4. Take the inverse Laplace transform of both sides. The LHS will simply be y, and you can use a table to look up the value for each piece of the RHS.

Partial fraction decomposition

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2. $$\frac{A}{(s-a)^{n+1}}$$
3. $$\frac{As+B}{(s-a)^2+b^2}$$
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When taking inverse Laplace transforms, these become:

1. Ae^{at}
2. $At^n e^{at}$
3. A linear combination of $e^{at}\cos(bt)$ and $e^{at}\sin(bt)$.
4. A linear combination of $e^{at}\cos(bt)$, $e^{at}\sin(bt)$, and their products by $t,\ t^2,\ \ldots,\ t^n$.

For all of these, it may be that $a=0$ so the exponential term is just 1.

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Now inverse Laplace transform:

$$y = -e^t \cos(2t) + \frac{3}{2}e^t \sin(2t) + e^t$$

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Should get $y = e^t + 2t^3e^t$.