# Math 302: The Laplace Transform, I

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Spring 2021

# A broad picture

A template used to solve problems in mathematics:

- Transfer your problem to a new problem in a new domain.
- Solve this problem in the new domain.
- Transfer the solution back to the original domain.

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Our next topic is the Laplace transform, used to solve differential equations by transfering from a time domain to a frequency domain.

# Improper integrals

The Laplace transform is defined in terms of improper integrals, so let's briefly review them.

- Intuitively, $\int_0^\infty f(t)\,\mathrm{d}t$ is the area of the region bounded by the two axes and the graph of $f(t)$.
- (It's a little more complicated to handle positive area above the axis versus negative area below the axis, but that's the gist.)
- Formally, this improper integral is defined as a certain limit:

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- If this integral converges for some value $s = s_0$, then it converges for all values $s > s_0$.

The idea: if $s > s_0$, then $e^{-st} < e^{-s_0 t}$ for all $t$. So the area for $s$ must be smaller than the area for $s_0$.

# The Laplace transform

Consider a function $f(t)$ in the time domain $t$ defined on $0 \leq t < \infty$.

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Examples:

- $\mathcal{L}[1](s) = \int_0^\infty e^{-st}\,\mathrm{d}t = \frac{1}{s}$
- $\mathcal{L}[t](s) = \int_0^\infty te^{-st}\,\mathrm{d}t = \frac{1}{s^2}$

This is what we computed two slides ago.

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- Let's rewrite this in base $e$ instead of base $x$:

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- Now substitute $-s = \log x$:

$$\int_0^\infty f(t)e^{-st} \, dt.$$

# Some properties of the Laplace transform

The Laplace transform is a linear operator.
That is,

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This is not difficult to check, and it comes from the fact that integration is a linear operator.

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There is a formula for the inverse Laplace transform, but it's based on complex integration, so I won't talk about it.

# Computing Laplace transforms by hand

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  by hand. Instead, you just remember $\frac{\mathrm{d}}{\mathrm{d}x} \sin x = \cos x$.

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We'll compute some examples by hand in lecture, but for homework you are encouraged to use a table of Laplace transforms, such as can be found on page 306 of your textbook.