Math 302: Integrating factors

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Spring 2021

Last time

On Monday, we talked about exact differential equations. If

$$P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y = 0$$

is exact, meaning $P_y = Q_x$, then you solve the integration by finding $f(x, y)$ so that $f_x = P$ and $f_y = Q$.

A couple examples

$$e^x \, dx + 2y \, dy = 0$$

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$$e^x \, \mathrm{d}x + 2y \, \mathrm{d}y = 0$$

$$\cos y \, \mathrm{d}x - (x \sin y - y^2) \, \mathrm{d}y = 0$$

Inexact differential equations

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- But what to do if you have an inexact differential equation?

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- But what to do if you have an inexact differential equation?

Multiply your ODE by something which makes it exact!

An example

$$(y^2 + y)\,\mathrm{d}x - x\,\mathrm{d}y = 0$$

is not exact.

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is not exact. But if we multiply both sides by $1/y^2 \ldots$

Integrating factors

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- On the previous slide, we saw that $1/y^2$ is an integrating factor for

$$(y^2 + y)\,dx - x\,dy = 0$$

Finding integrating factors

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This maybe doesn't look useful for finding out what the integrating factor m has to be, but if it's in a special form then we can work it out.

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If the integrating factor m only depends on one variable, then we can explicitly work out what it has to be. Let's see the $m = m(x)$ case:

$$\frac{\partial}{\partial y}(m(x)P(x,y)) = \frac{\partial}{\partial x}(m(x)Q(x,y))$$

$$m(x)P_y = Q_x(x,y)m(x) + Q(x,y)\frac{\mathrm{d}m(x)}{\mathrm{d}x}$$

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Rearrange to:

$$\frac{\mathrm{d}m}{m} = \frac{P_y - Q_x}{Q}\,\mathrm{d}x$$

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Rearrange to:

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The LHS only depends on x, so the same must be true of the RHS. So solving this separable differential equation in m and x gives

$$m(x) = \exp\left(\int \frac{P_y - Q_x}{Q}\,\mathrm{d}x\right).$$

An example

$$xy\,\mathrm{d}x + (1 + x^2)\,\mathrm{d}y = 0$$

A general rule

Suppose you have an inexact differential equation

$$P(x, y)\,dx + Q(x, y)\,dy = 0.$$

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- If neither of these is the case, you may still be able to find an integrating factor, but it's harder. The book has a few more special cases, but we won't cover them.