Math 302: Differential equations with homogeneous coefficients

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A differential equation

$$(x + y)\,\mathrm{d}x + (x + x^2/y)\,\mathrm{d}y = 0$$

Homogeneous functions

A function $f(x, y)$ in two variables x and y is homogeneous of order α if it can be written as

$$f(x, y) = x^\alpha g(y/x) \qquad \text{or} \qquad f(x, y) = y^\alpha g(x/y),$$

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- And it's probably not clear why a function like this would be called homogenous.
- Here's an equivalent definition:

$f(x, y)$ is homogenous of order α if for any real number λ we have $f(\lambda x, \lambda y) = \lambda^{\alpha} f(x, y)$.

That is, if you scale the inputs by a constant, then you scale the output by a power of that constant.

Let's quickly check the two definitions are equivalent

$$f(x, y) = x^\alpha g(y/x) \qquad \text{versus} \qquad f(\lambda x, \lambda y) = \lambda^\alpha f(x, y)$$

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- Any linear function $\ell(x, y) = ax + by$ is homogenous of order 1.
- A monomial $x^a y^b$ is homogenous of order $a + b$.
- More generally, any sum of monomials of the same degree α is homogeneous of order α.

Differential equations with homogenous coefficients

A differential equation of the form

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- If a differential equation has homogenous coefficients, then we can solve it by doing a substitution to turn it into a separable differential equation.
- Namely, use the substitution

  $$u = \frac{y}{x}, \qquad y = ux, \qquad \mathrm{d}y = u\,\mathrm{d}x + x\,\mathrm{d}u$$

  or, alternatively, the substitution

  $$u = \frac{x}{y}, \qquad x = uy, \qquad \mathrm{d}x = u\,\mathrm{d}y + y\,\mathrm{d}u.$$

Another example

$$xy' - y - x\sin(y/x) = 0$$