Math 302: Existence and uniqueness of solutions

Kameryn J Williams

University of Hawaiʻi at Mānoa

Spring 2021

Solutions to differential equations

- All semester we've been learning methods to solve differential equations.

Solutions to differential equations

- All semester we've been learning methods to solve differential equations.
- We haven't exhausted the topic. You could spend semesters learning more and more methods to solve more and more kinds of differential equations.

Solutions to differential equations

- All semester we've been learning methods to solve differential equations.
- We haven't exhausted the topic. You could spend semesters learning more and more methods to solve more and more kinds of differential equations.
- For the last topic this semester, let's step away from calculations to ask a theoretical question:
- When, in generality, can we say that a differential equation has solutions? When are solutions uniquely identified?

Solutions to differential equations

- All semester we've been learning methods to solve differential equations.
- We haven't exhausted the topic. You could spend semesters learning more and more methods to solve more and more kinds of differential equations.
- For the last topic this semester, let's step away from calculations to ask a theoretical question:
- When, in generality, can we say that a differential equation has solutions? When are solutions uniquely identified?

I won't go into every gritty detail. Rather, the goal is to give you a view of the broad ideas, and some techniques and tricks that are used to reason abstractly about differential equations.

Linear differential equations

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = b(x)$$

Linear differential equations

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = b(x)$$

A trick: Let's transform this from a high-order problem in one function y to a first-order problem in many functions $y_1, \ldots, y_n$:

Linear differential equations

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = b(x)$$

A trick: Let's transform this from a high-order problem in one function y to a first-order problem in many functions $y_1, \ldots, y_n$:

$$\begin{aligned} y_1 &= y \\ y_2 &= y' = y_1' \\ &\vdots \\ y_n &= y^{(n-1)} = y_{n-1}' \end{aligned}$$

Linear differential equations

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = b(x)$$

A trick: Let's transform this from a high-order problem in one function y to a first-order problem in many functions $y_1, \ldots, y_n$:

$$\begin{aligned} y_1 &= y \\ y_2 &= y' = y_1' \\ &\vdots \\ y_n &= y^{(n-1)} = y_{n-1}' \end{aligned}$$

We can turn our original equation into a system of equations about the derivatives of the y_i.

Linear differential equations

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = b(x)$$

A trick: Let's transform this from a high-order problem in one function y to a first-order problem in many functions $y_1, \dots, y_n$:

$$\begin{aligned} y_1 &= y \\ y_2 &= y' = y_1' \\ &\vdots \\ y_n &= y^{(n-1)} = y_{n-1}' \end{aligned}$$

We can turn our original equation into a system of equations about the derivatives of the y_i.

$$\begin{aligned} y_n' &= -a_{n-1}y_n - \cdots - a_1y_2 - a_0y_1 + b \\ y_{n-1}' &= y_n \\ &\vdots \\ y_1' &= y_2 \end{aligned}$$

So solving this system of equations gives us a solution to our original equation.

Linear differential equations

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = b(x)$$

A trick: Let's transform this from a high-order problem in one function y to a first-order problem in many functions $y_1, \ldots, y_n$:

$$\begin{aligned} y_1 &= y \\ y_2 &= y' = y_1' \\ &\vdots \\ y_n &= y^{(n-1)} = y_{n-1}' \end{aligned}$$

We can turn our original equation into a system of equations about the derivatives of the y_i.

$$\begin{aligned} y_n' &= -a_{n-1}y_n - \cdots - a_1 y_2 - a_0 y_1 + b \\ y_{n-1}' &= y_n \\ &\vdots \\ y_1' &= y_2 \end{aligned}$$

So solving this system of equations gives us a solution to our original equation.

We've turned our problem into a new problem. That's kinda like progress!

Solving systems of equations

We turned our nth-order differential equation about an unknown function y into a system of equations. Namely, we have

- n equations
- about n unknown functions $y_1, \ldots, y_n$
- where the ith equation describes y_i' in terms of $y_1, \ldots, y_n$, and x.

Solving systems of equations

We turned our nth-order differential equation about an unknown function y into a system of equations. Namely, we have

- n equations
- about n unknown functions $y_1, \ldots, y_n$
- where the ith equation describes y_i' in terms of $y_1, \ldots, y_n$, and x.

Let's look at this in a bit more generality.

$$\begin{aligned} y_1' &= F_1(x, y_1, \ldots, y_n) \\ &\vdots \\ y_n' &= F_n(x, y_1, \ldots, y_n) \end{aligned}$$

Solving systems of equations

We turned our nth-order differential equation about an unknown function y into a system of equations. Namely, we have

- n equations
- about n unknown functions $y_1, \ldots, y_n$
- where the ith equation describes y_i' in terms of $y_1, \ldots, y_n$, and x.

Let's look at this in a bit more generality.

$$\begin{aligned} y_1' &= F_1(x, y_1, \ldots, y_n) \\ &\vdots \\ y_n' &= F_n(x, y_1, \ldots, y_n) \end{aligned}$$

Think back to the beginning of the semester when we looked at slope fields.

- We had an equation $y' = F(x, y)$, which we could think of as assigning a slope to every point in the plane.

Solving systems of equations

We turned our nth-order differential equation about an unknown function y into a system of equations. Namely, we have

- n equations
- about n unknown functions $y_1, \ldots, y_n$
- where the ith equation describes y_i' in terms of $y_1, \ldots, y_n$, and x.

Let's look at this in a bit more generality.

$$\begin{aligned} y_1' &= F_1(x, y_1, \ldots, y_n) \\ &\vdots \\ y_n' &= F_n(x, y_1, \ldots, y_n) \end{aligned}$$

Think back to the beginning of the semester when we looked at slope fields.

- We had an equation $y' = F(x, y)$, which we could think of as assigning a slope to every point in the plane.
- What we have here is a higher dimensional version of this. Rather than one slope in 2-dimensional space, we look at n slopes in $(n+1)$-dimensional space.

Solving systems of equations

We turned our nth-order differential equation about an unknown function y into a system of equations. Namely, we have

- n equations
- about n unknown functions $y_1, \dots, y_n$
- where the ith equation describes y_i' in terms of $y_1, \dots, y_n$, and x.

Let's look at this in a bit more generality.

$$\begin{aligned} y_1' &= F_1(x, y_1, \dots, y_n) \\ &\vdots \\ y_n' &= F_n(x, y_1, \dots, y_n) \end{aligned}$$

Think back to the beginning of the semester when we looked at slope fields.

- We had an equation $y' = F(x, y)$, which we could think of as assigning a slope to every point in the plane.
- What we have here is a higher dimensional version of this. Rather than one slope in 2-dimensional space, we look at n slopes in $(n+1)$-dimensional space.

If we want to think about how to solve this, it makes sense to start with the simplest case, where we have only one unknown function.

Solving $y' = F(x, y)$

- Suppose for now that F is continuous in a rectangle R: $a_0 \leq x \leq a_1$ and $b_0 \leq y \leq b_1$.
- (We'll see later that we actually have to assume a little bit more about F.)
- Fix a point (x_0, y_0) in this rectangle, and suppose we have have the initial condition $y(x_0) = y_0$.

Solving $y' = F(x, y)$

A trick: Turn the differential equation into an integral equation.

Integrate both sides, starting at x_0, get:

$$y(x) - y(x_0) = \int_{x_0}^{x} F(t, y(t))\, dt.$$

- Suppose for now that F is continuous in a rectangle R: $a_0 \leq x \leq a_1$ and $b_0 \leq y \leq b_1$.
- (We'll see later that we actually have to assume a little bit more about F.)
- Fix a point (x_0, y_0) in this rectangle, and suppose we have have the initial condition $y(x_0) = y_0$.

Solving $y' = F(x, y)$

- Suppose for now that F is continuous in a rectangle R: $a_0 \leq x \leq a_1$ and $b_0 \leq y \leq b_1$.
- (We'll see later that we actually have to assume a little bit more about F.)
- Fix a point (x_0, y_0) in this rectangle, and suppose we have have the initial condition $y(x_0) = y_0$.

A trick: Turn the differential equation into an integral equation.

Integrate both sides, starting at x_0, get:

$$y(x) - y(x_0) = \int_{x_0}^{x} F(t, y(t))\, dt.$$

Why?

Solving $y' = F(x, y)$

- Suppose for now that F is continuous in a rectangle R: $a_0 \le x \le a_1$ and $b_0 \le y \le b_1$.
- (We'll see later that we actually have to assume a little bit more about F.)
- Fix a point (x_0, y_0) in this rectangle, and suppose we have have the initial condition $y(x_0) = y_0$.

A trick: Turn the differential equation into an integral equation.

Integrate both sides, starting at x_0, get:

$$y(x) - y(x_0) = \int_{x_0}^{x} F(t, y(t))\,dt.$$

Why?

- Derivatives are easier to compute than integrals.
- But for theoretical uses, integrals are better behaved.
- Since we want a theoretical result rather than calculations, this makes things easier.

An integral equation

$$y(x) = y_0 + \int_{x_0}^{x} F(t, y(t))\, \mathrm{d}t$$

An integral equation

$$y(x) = y_0 + \int_{x_0}^{x} F(t, y(t))\, dt$$

A stupid idea: What if we guess that $y(x) = y_0$?

An integral equation

$$y(x) = y_0 + \int_{x_0}^{x} F(t, y(t))\,\mathrm{d}t$$

A stupid idea: What if we guess that $y(x) = y_0$?

This obviously won't work, because in general

$$y_0 + \int_{x_0}^{x} F(t, y_0)\,\mathrm{d}t$$

is some new function, not the constant function y_0.

An integral equation

$$y(x) = y_0 + \int_{x_0}^{x} F(t, y(t))\,\mathrm{d}t$$

A stupid idea: What if we guess that $y(x) = y_0$?

This obviously won't work, because in general

$$y_0 + \int_{x_0}^{x} F(t, y_0)\,\mathrm{d}t$$

is some new function, not the constant function y_0.

But still! What if we took the output of this process, and did it again? And then kept repeating it, and hoped that the error got smaller and smaller.

An integral equation

$$y(x) = y_0 + \int_{x_0}^{x} F(t, y(t))\,\mathrm{d}t$$

A stupid idea: What if we guess that $y(x) = y_0$?

This obviously won't work, because in general

$$y_0 + \int_{x_0}^{x} F(t, y_0)\,\mathrm{d}t$$

is some new function, not the constant function y_0.

But still! What if we took the output of this process, and did it again? And then kept repeating it, and hoped that the error got smaller and smaller.

$$y_0(x) = y_0$$

$$y_1(x) = y_0 + \int_{x_0}^{x} F(t, y_0(t))\,\mathrm{d}t$$

$$\vdots$$

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} F(t, y_n(t))\,\mathrm{d}t$$

$$\vdots$$

An integral equation

$$y(x) = y_0 + \int_{x_0}^{x} F(t, y(t))\,\mathrm{d}t$$

A stupid idea: What if we guess that $y(x) = y_0$?

This obviously won't work, because in general

$$y_0 + \int_{x_0}^{x} F(t, y_0)\,\mathrm{d}t$$

is some new function, not the constant function y_0.

But still! What if we took the output of this process, and did it again? And then kept repeating it, and hoped that the error got smaller and smaller.

$$y_0(x) = y_0$$

$$y_1(x) = y_0 + \int_{x_0}^{x} F(t, y_0(t))\,\mathrm{d}t$$

$$\vdots$$

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} F(t, y_n(t))\,\mathrm{d}t$$

$$\vdots$$

Spoiler: This stupid idea will work out (with an extra assumption on F). The error gets smaller and smaller, so these Picard approximations converge to the true solution.

An example integral equation: $F(x, y) = x + y$ and $x_0 = y_0 = 0$

$$y(x) = y_0 + \int_{x_0}^{x} F(t, y(t))\,\mathrm{d}t$$

An example integral equation: $F(x,y) = x + y$ and $x_0 = y_0 = 0$

$$y(x) = \int_0^x t + y(t)\,\mathrm{d}t$$

A recap

- We want to prove that a linear differential equation

$$y^{(n)} + \cdots + a_1(x)y' + a_0(x)y = b(x)$$

has solutions.

A recap

- We want to prove that a linear differential equation

$$y^{(n)} + \cdots + a_1(x)y' + a_0(x)y = b(x)$$

  has solutions.
- We reduced this problem about up to nth derivatives of a single function to a problem about 1st derivatives of n many functions.

A recap

- We want to prove that a linear differential equation

$$y^{(n)} + \cdots + a_1(x)y' + a_0(x)y = b(x)$$

  has solutions.
- We reduced this problem about up to nth derivatives of a single function to a problem about 1st derivatives of n many functions.
- We're first trying to solve the 1 dimensional version of this problem, before looking at the general case.

A recap

- We want to prove that a linear differential equation

$$y^{(n)} + \cdots + a_1(x)y' + a_0(x)y = b(x)$$

  has solutions.
- We reduced this problem about up to nth derivatives of a single function to a problem about 1st derivatives of n many functions.
- We're first trying to solve the 1 dimensional version of this problem, before looking at the general case.
- We turned the first-order differential equation $y' = F(x, y)$ into an integral equation.

A recap

- We want to prove that a linear differential equation

$$y^{(n)} + \cdots + a_1(x)y' + a_0(x)y = b(x)$$

  has solutions.
- We reduced this problem about up to nth derivatives of a single function to a problem about 1st derivatives of n many functions.
- We're first trying to solve the 1 dimensional version of this problem, before looking at the general case.
- We turned the first-order differential equation $y' = F(x, y)$ into an integral equation.
- We want to use Picard's method to find a solution to this integral equation.

Picard's method

$$y(x) = y_0 + \int_{x_0}^{x} F(t, y(t))\,\mathrm{d}t$$

We want to solve this equation. The idea is to get the solution from a sequence of better and better approximations:

$$\begin{aligned}
y_0(x) &= y_0 \\
y_1(x) &= y_0 + \int_{x_0}^{x} F(t, y_0(t))\,\mathrm{d}t \\
&\vdots \\
y_{n+1}(x) &= y_0 + \int_{x_0}^{x} F(t, y_n(t))\,\mathrm{d}t \\
&\vdots
\end{aligned}$$

The solution $y(x)$ will be the limit of this sequence.

Picard's method

$$y(x) = y_0 + \int_{x_0}^{x} F(t, y(t))\, dt$$

We want to solve this equation. The idea is to get the solution from a sequence of better and better approximations:

$$y_0(x) = y_0$$

$$y_1(x) = y_0 + \int_{x_0}^{x} F(t, y_0(t))\, dt$$

$$\vdots$$

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} F(t, y_n(t))\, dt$$

$$\vdots$$

The solution $y(x)$ will be the limit of this sequence.

- In fact, it doesn't matter what we start with as our first guess $y_0(x)$. It can be any continuous function and we will still converge to the correct solution in the limit.

Picard's method

$$y(x) = y_0 + \int_{x_0}^{x} F(t, y(t))\,\mathrm{d}t$$

We want to solve this equation. The idea is to get the solution from a sequence of better and better approximations:

$$\begin{aligned} y_0(x) &= y_0 \\ y_1(x) &= y_0 + \int_{x_0}^{x} F(t, y_0(t))\,\mathrm{d}t \\ &\vdots \\ y_{n+1}(x) &= y_0 + \int_{x_0}^{x} F(t, y_n(t))\,\mathrm{d}t \\ &\vdots \end{aligned}$$

The solution $y(x)$ will be the limit of this sequence.

- In fact, it doesn't matter what we start with as our first guess $y_0(x)$. It can be any continuous function and we will still converge to the correct solution in the limit.

Explaining just why this works will take some time.

Let's start by talking about convergence of sequences of functions.

Convergence of sequences of functions

Convergence and limits are a subtle topic.
We'll have to be a bit more careful then was necessary for the calculus sequence.

Convergence of sequences of functions

Convergence and limits are a subtle topic.
We'll have to be a bit more careful then was necessary for the calculus sequence.

- A sequence of functions $y_0, y_1, \ldots$ on a common domain D converge to a function y if $\lim_{n\to\infty} y_n(x) = y(x)$ for every x in D.

Convergence of sequences of functions

Convergence and limits are a subtle topic.
We'll have to be a bit more careful then was necessary for the calculus sequence.

- A sequence of functions $y_0, y_1, \ldots$ on a common domain D converge to a function y if $\lim_{n\to\infty} y_n(x) = y(x)$ for every x in D.
  - More precisely, for any point x in D and any error $\varepsilon > 0$, if $n > N$ for some large enough N then $|y_n(x) - y(x)| < \varepsilon$.

Convergence of sequences of functions

Convergence and limits are a subtle topic.
We'll have to be a bit more careful then was necessary for the calculus sequence.

- A sequence of functions $y_0, y_1, \ldots$ on a common domain D converge to a function y if $\lim_{n\to\infty} y_n(x) = y(x)$ for every x in D.
  - More precisely, for any point x in D and any error $\varepsilon > 0$, if $n > N$ for some large enough N then $|y_n(x) - y(x)| < \varepsilon$.
- The functions $y_0, y_1, \ldots$ converge uniformly if you can use the same error for *any* choice of x.

Convergence of sequences of functions

Convergence and limits are a subtle topic. We'll have to be a bit more careful then was necessary for the calculus sequence.

- A sequence of functions $y_0, y_1, \ldots$ on a common domain D converge to a function y if $\lim_{n\to\infty} y_n(x) = y(x)$ for every x in D.
  - More precisely, for any point x in D and any error $\varepsilon > 0$, if $n > N$ for some large enough N then $|y_n(x) - y(x)| < \varepsilon$.
- The functions $y_0, y_1, \ldots$ converge uniformly if you can use the same error for *any* choice of x.
  - That is, for any error $\varepsilon > 0$ if $n > N$ for some large enough N then $|y_n(x) - y(x)| < \varepsilon$ for any choice of x.

Convergence of sequences of functions

Convergence and limits are a subtle topic. We'll have to be a bit more careful then was necessary for the calculus sequence.

- A sequence of functions $y_0, y_1, \ldots$ on a common domain D converge to a function y if $\lim_{n\to\infty} y_n(x) = y(x)$ for every x in D.
  - More precisely, for any point x in D and any error $\varepsilon > 0$, if $n > N$ for some large enough N then $|y_n(x) - y(x)| < \varepsilon$.
- The functions $y_0, y_1, \ldots$ converge uniformly if you can use the same error for *any* choice of x.
  - That is, for any error $\varepsilon > 0$ if $n > N$ for some large enough N then $|y_n(x) - y(x)| < \varepsilon$ for any choice of x.

Uniform convergence is better behaved than mere convergence.

Convergence of sequences of functions

Convergence and limits are a subtle topic. We'll have to be a bit more careful then was necessary for the calculus sequence.

- A sequence of functions $y_0, y_1, \ldots$ on a common domain D converge to a function y if $\lim_{n\to\infty} y_n(x) = y(x)$ for every x in D.
  - More precisely, for any point x in D and any error $\varepsilon > 0$, if $n > N$ for some large enough N then $|y_n(x) - y(x)| < \varepsilon$.
- The functions $y_0, y_1, \ldots$ converge uniformly if you can use the same error for *any* choice of x.
  - That is, for any error $\varepsilon > 0$ if $n > N$ for some large enough N then $|y_n(x) - y(x)| < \varepsilon$ for any choice of x.

Uniform convergence is better behaved than mere convergence.

For example:

- If the functions $y_0, y_1, \ldots$ are all continuous on D and converge uniformly to y, then y is continuous on D.
- If they merely converge to y, then y need not be continuous.

Convergence of sequences of functions

Convergence and limits are a subtle topic. We'll have to be a bit more careful then was necessary for the calculus sequence.

- A sequence of functions $y_0, y_1, \ldots$ on a common domain D converge to a function y if $\lim_{n\to\infty} y_n(x) = y(x)$ for every x in D.
  - More precisely, for any point x in D and any error $\varepsilon > 0$, if $n > N$ for some large enough N then $|y_n(x) - y(x)| < \varepsilon$.
- The functions $y_0, y_1, \ldots$ converge uniformly if you can use the same error for *any* choice of x.
  - That is, for any error $\varepsilon > 0$ if $n > N$ for some large enough N then $|y_n(x) - y(x)| < \varepsilon$ for any choice of x.

Uniform convergence is better behaved than mere convergence.

For example:

- If the functions $y_0, y_1, \ldots$ are all continuous on D and converge uniformly to y, then y is continuous on D.
- If they merely converge to y, then y need not be continuous.
  - The functions $1, x^2, x^3, \ldots$ are continuous on $[0, 1]$ and converge (nonuniformly) to the discontinuous function which is 0 when $x < 1$ and 1 when $x = 1$.

Convergence of sequences of functions

Convergence and limits are a subtle topic. We'll have to be a bit more careful then was necessary for the calculus sequence.

- A sequence of functions $y_0, y_1, \ldots$ on a common domain D converge to a function y if $\lim_{n\to\infty} y_n(x) = y(x)$ for every x in D.
  - More precisely, for any point x in D and any error $\varepsilon > 0$, if $n > N$ for some large enough N then $|y_n(x) - y(x)| < \varepsilon$.
- The functions $y_0, y_1, \ldots$ converge uniformly if you can use the same error for *any* choice of x.
  - That is, for any error $\varepsilon > 0$ if $n > N$ for some large enough N then $|y_n(x) - y(x)| < \varepsilon$ for any choice of x.

Uniform convergence is better behaved than mere convergence.

For example:

- If the functions $y_0, y_1, \ldots$ are all continuous on D and converge uniformly to y, then y is continuous on D.
- If they merely converge to y, then y need not be continuous.
  - The functions $1, x^2, x^3, \ldots$ are continuous on $[0, 1]$ and converge (nonuniformly) to the discontinuous function which is 0 when $x < 1$ and 1 when $x = 1$.

A lot of things you might hope convergence is enough for actually need uniform convergence.

Uniform convergence and integration

Suppose the functions $y_0, y_1, \ldots$ converge uniformly to y on a domain which includes the interval $[a, b]$.

- Then, you can swap the order of integration and taking a limit:

$$\lim_{n\to\infty} \int_a^b y_n(x)\,dx = \int_a^b \underbrace{\lim_{n\to\infty} y_n(x)}_{=y(x)}\,dx.$$

Uniform convergence and integration

Suppose the functions $y_0, y_1, \ldots$ converge uniformly to y on a domain which includes the interval $[a, b]$.

- Then, you can swap the order of integration and taking a limit:

$$\lim_{n\to\infty} \int_a^b y_n(x)\,dx = \int_a^b \underbrace{\lim_{n\to\infty} y_n(x)}_{=y(x)}\,dx.$$

We want to apply this to our sequence of Picard approximations:

$$\begin{aligned}\lim_{n\to\infty} y_n(x) &= \lim_{n\to\infty} \left(y_0 + \int_{x_0}^x F(t, y_{n-1}(t))\,dt \right) \\ &= y_0 + \int_{x_0}^x \lim_{n\to\infty} F(t, y_{n-1}(t))\,dt\end{aligned}$$

Uniform convergence and integration

Suppose the functions $y_0, y_1, \dots$ converge uniformly to y on a domain which includes the interval $[a, b]$.

- Then, you can swap the order of integration and taking a limit:

$$\lim_{n\to\infty} \int_a^b y_n(x)\,\mathrm{d}x = \int_a^b \underbrace{\lim_{n\to\infty} y_n(x)}_{=y(x)}\,\mathrm{d}x.$$

We want to apply this to our sequence of Picard approximations:

$$\begin{aligned}\lim_{n\to\infty} y_n(x) &= \lim_{n\to\infty}\left(y_0 + \int_{x_0}^{x} F(t, y_{n-1}(t))\,\mathrm{d}t\right)\\ &= y_0 + \int_{x_0}^{x} \lim_{n\to\infty} F(t, y_{n-1}(t))\,\mathrm{d}t\end{aligned}$$

To be able to bring the limit inside the integral we need that the sequence

$$F(t, y_0(t)), F(t, y_1(t)), \dots, F(t, y_n(t)), \dots$$

converges uniformly in the interval $[x_0, x]$.

One way we might get convergence

Maybe we can show that the differences $y_{n+1} - y_n$ get uniformly smaller, so that the telescoping series

$$(y_0 - y_1) + (y_1 - y_2) + (y_2 - y_3) + \cdots$$

converges uniformly.

One way we might get convergence

Maybe we can show that the differences $y_{n+1} - y_n$ get uniformly smaller, so that the telescoping series

$$(y_0 - y_1) + (y_1 - y_2) + (y_2 - y_3) + \cdots$$

converges uniformly.

Let's look at this and think aboout what we might need.

$$|y_1 - y_0| = \left| \int_{x_0}^{x} F(t, y_0)\,\mathrm{d}t \right| \leq \int_{x_0}^{x} |F(t, y_0)|\,\mathrm{d}t$$

One way we might get convergence

Maybe we can show that the differences $y_{n+1} - y_n$ get uniformly smaller, so that the telescoping series

$$(y_0 - y_1) + (y_1 - y_2) + (y_2 - y_3) + \cdots$$

converges uniformly.

Let's look at this and think aboout what we might need.

$$|y_1 - y_0| = \left| \int_{x_0}^{x} F(t, y_0)\, \mathrm{d}t \right| \leq \int_{x_0}^{x} |F(t, y_0)|\, \mathrm{d}t$$

Because F is a continuous function on a bounded domain, $|F(x, y)|$ has some maximum value M.

$$|y_1 - y_0| \leq \int_{x_0}^{x} M\, \mathrm{d}t = M\, |x - x_0|$$

One way we might get convergence

Maybe we can show that the differences $y_{n+1} - y_n$ get uniformly smaller, so that the telescoping series

$$(y_0 - y_1) + (y_1 - y_2) + (y_2 - y_3) + \cdots$$

converges uniformly.

Let's look at this and think aboout what we might need.

$$|y_1 - y_0| = \left| \int_{x_0}^{x} F(t, y_0)\,\mathrm{d}t \right| \leq \int_{x_0}^{x} |F(t, y_0)|\,\mathrm{d}t$$

Because F is a continuous function on a bounded domain, $|F(x, y)|$ has some maximum value M.

$$|y_1 - y_0| \leq \int_{x_0}^{x} M\,\mathrm{d}t = M\,|x - x_0|$$

Since we're looking at a bounded domain for x, we have some upper bound D for $|x - x_0|$, so we can conclude

$$|y_1(x) - y_0(x)| \leq MD$$

for all x in our domain.

One way we might get convergence

Maybe we can show that the differences $y_{n+1} - y_n$ get uniformly smaller, so that the telescoping series

$$(y_0 - y_1) + (y_1 - y_2) + (y_2 - y_3) + \cdots$$

converges uniformly.

Let's look at this and think aboout what we might need.

$$|y_1 - y_0| = \left| \int_{x_0}^{x} F(t, y_0)\, \mathrm{d}t \right| \leq \int_{x_0}^{x} |F(t, y_0)|\, \mathrm{d}t$$

Because F is a continuous function on a bounded domain, $|F(x, y)|$ has some maximum value M.

$$|y_1 - y_0| \leq \int_{x_0}^{x} M\, \mathrm{d}t = M\,|x - x_0|$$

Since we're looking at a bounded domain for x, we have some upper bound D for $|x - x_0|$, so we can conclude

$$|y_1(x) - y_0(x)| \leq MD$$

for all x in our domain.

That's a start. We got the first step. But later ones are gonna be harder...

Why is it harder?

We want to compute a bound on $|y_{n+1} - y_n|$.
Let's start

$$\begin{aligned} |y_{n+1} - y_n| &= \left| \int_{x_0}^{x} F(t, y_{n+1}(t)) - F(t, y_n(t)) \, \mathrm{d}t \right| \\ &\leq \int_{x_0}^{x} |F(t, y_{n+1}(t)) - F(t, y_n(t))| \, \mathrm{d}t \end{aligned}$$

Why is it harder?

We want to compute a bound on $|y_{n+1} - y_n|$.
Let's start

$$
\begin{aligned}
|y_{n+1} - y_n| &= \left| \int_{x_0}^{x} F(t, y_{n+1}(t)) - F(t, y_n(t))\, \mathrm{d}t \right| \\
&\leq \int_{x_0}^{x} |F(t, y_{n+1}(t)) - F(t, y_n(t))|\, \mathrm{d}t
\end{aligned}
$$

We know that $0 \leq |F(x, y)| \leq M$ on our domain, so the distance between $F(t, y_{n+1}(t))$ and $F(t, y_n(t))$ is at most $2M$.

Why is it harder?

We want to compute a bound on $|y_{n+1} - y_n|$.
Let's start

$$\begin{aligned} |y_{n+1} - y_n| &= \left| \int_{x_0}^{x} F(t, y_{n+1}(t)) - F(t, y_n(t)) \, \mathrm{d}t \right| \\ &\leq \int_{x_0}^{x} |F(t, y_{n+1}(t)) - F(t, y_n(t))| \, \mathrm{d}t \end{aligned}$$

We know that $0 \leq |F(x, y)| \leq M$ on our domain, so the distance between $F(t, y_{n+1}(t))$ and $F(t, y_n(t))$ is at most $2M$. So we get:

$$|y_{n+1} - y_n| \int_{x_0}^{x} 2M \, \mathrm{d}t \leq 2MD.$$

Why is it harder?

We want to compute a bound on $|y_{n+1} - y_n|$. Let's start

$$|y_{n+1} - y_n| = \left| \int_{x_0}^{x} F(t, y_{n+1}(t)) - F(t, y_n(t)) \, dt \right|$$
$$\leq \int_{x_0}^{x} |F(t, y_{n+1}(t)) - F(t, y_n(t))| \, dt$$

We know that $0 \leq |F(x, y)| \leq M$ on our domain, so the distance between $F(t, y_{n+1}(t))$ and $F(t, y_n(t))$ is at most $2M$. So we get:

$$|y_{n+1} - y_n| \int_{x_0}^{x} 2M \, dt \leq 2MD.$$

This gives us a bound on the telescoping series

$$(y_0 - y_1) + (y_1 - y_2) + (y_2 - y_3) + \cdots$$

Why is it harder?

We want to compute a bound on $|y_{n+1} - y_n|$. Let's start

$$|y_{n+1} - y_n| = \left| \int_{x_0}^{x} F(t, y_{n+1}(t)) - F(t, y_n(t))\, \mathrm{d}t \right|$$
$$\leq \int_{x_0}^{x} |F(t, y_{n+1}(t)) - F(t, y_n(t))|\, \mathrm{d}t$$

We know that $0 \leq |F(x,y)| \leq M$ on our domain, so the distance between $F(t, y_{n+1}(t))$ and $F(t, y_n(t))$ is at most $2M$. So we get:

$$|y_{n+1} - y_n| \int_{x_0}^{x} 2M\, \mathrm{d}t \leq 2MD.$$

This gives us a bound on the telescoping series

$$(y_0 - y_1) + (y_1 - y_2) + (y_2 - y_3) + \cdots$$

Namely, its absolute value is bounded by:

$$MD + 2MD + 2MD + \cdots$$

Why is it harder?

We want to compute a bound on $|y_{n+1} - y_n|$. Let's start

$$|y_{n+1} - y_n| = \left| \int_{x_0}^{x} F(t, y_{n+1}(t)) - F(t, y_n(t))\, \mathrm{d}t \right|$$
$$\leq \int_{x_0}^{x} |F(t, y_{n+1}(t)) - F(t, y_n(t))|\, \mathrm{d}t$$

We know that $0 \leq |F(x, y)| \leq M$ on our domain, so the distance between $F(t, y_{n+1}(t))$ and $F(t, y_n(t))$ is at most $2M$. So we get:

$$|y_{n+1} - y_n| \int_{x_0}^{x} 2M\, \mathrm{d}t \leq 2MD.$$

This gives us a bound on the telescoping series

$$(y_0 - y_1) + (y_1 - y_2) + (y_2 - y_3) + \cdots$$

Namely, its absolute value is bounded by:

$$MD + 2MD + 2MD + \cdots$$

So we've seen it is $\leq \infty$. That's not helpful.

Why is it harder?

We want to compute a bound on $|y_{n+1} - y_n|$. Let's start

$$|y_{n+1} - y_n| = \left| \int_{x_0}^{x} F(t, y_{n+1}(t)) - F(t, y_n(t))\, dt \right|$$
$$\leq \int_{x_0}^{x} |F(t, y_{n+1}(t)) - F(t, y_n(t))|\ dt$$

We know that $0 \leq |F(x, y)| \leq M$ on our domain, so the distance between $F(t, y_{n+1}(t))$ and $F(t, y_n(t))$ is at most $2M$. So we get:

$$|y_{n+1} - y_n| \int_{x_0}^{x} 2M\, dt \leq 2MD.$$

This gives us a bound on the telescoping series

$$(y_0 - y_1) + (y_1 - y_2) + (y_2 - y_3) + \cdots$$

Namely, its absolute value is bounded by:

$$MD + 2MD + 2MD + \cdots$$

So we've seen it is $\leq \infty$. That's not helpful.

We need to do better.

Why is it harder?

We want to compute a bound on $|y_{n+1} - y_n|$. Let's start

$$|y_{n+1} - y_n| = \left| \int_{x_0}^{x} F(t, y_{n+1}(t)) - F(t, y_n(t))\,\mathrm{d}t \right|$$
$$\leq \int_{x_0}^{x} |F(t, y_{n+1}(t)) - F(t, y_n(t))|\,\mathrm{d}t$$

We know that $0 \leq |F(x, y)| \leq M$ on our domain, so the distance between $F(t, y_{n+1}(t))$ and $F(t, y_n(t))$ is at most $2M$. So we get:

$$|y_{n+1} - y_n| \int_{x_0}^{x} 2M\,\mathrm{d}t \leq 2MD.$$

This gives us a bound on the telescoping series

$$(y_0 - y_1) + (y_1 - y_2) + (y_2 - y_3) + \cdots$$

Namely, its absolute value is bounded by:

$$MD + 2MD + 2MD + \cdots$$

So we've seen it is $\leq \infty$. That's not helpful.

We need to do better.

The trouble: If F changes values really quickly, we can't get a better bound on the distance between $F(x_0, y_0)$ and $F(x_1, y_1)$ for two points. So we need to assume more about F.

This is a good place to step away from the weeds to handwave

This is a good place to step away from the weeds to handwave

A condition on F that makes this work is to assume that

$$|F(t, y_a) - F(t, y_b)| \leq L\,|y_a - y_b|$$

for some fixed bound L which works for all points y_a, y_b. This condition says that F can change, but it cannot change too quickly.

(This is known as a Lipschitz condition.)

This is a good place to step away from the weeds to handwave

A condition on F that makes this work is to assume that

$$|F(t, y_a) - F(t, y_b)| \leq L\,|y_a - y_b|$$

for some fixed bound L which works for all points y_a, y_b. This condition says that F can change, but it cannot change too quickly.

(This is known as a Lipschitz condition.)

With this extra assumption, we can compute a better bound:

$$|y_{n+1} - y_n| \leq ML^n \frac{D^{n+1}}{(n+1)!}$$

This is a good place to step away from the weeds to handwave

A condition on F that makes this work is to assume that

$$|F(t, y_a) - F(t, y_b)| \leq L\,|y_a - y_b|$$

for some fixed bound L which works for all points y_a, y_b. This condition says that F can change, but it cannot change too quickly.

(This is known as a Lipschitz condition.)

With this extra assumption, we can compute a better bound:

$$|y_{n+1} - y_n| \leq ML^n \frac{D^{n+1}}{(n+1)!}$$

Putting these together we get a bound for the telescoping series

$$(y_0 - y_1) + (y_1 - y_2) + (y_2 - y_3) + \cdots$$

Namely, its magnitude is bounded by:

$$y_0 + \frac{M}{L}\left(\frac{LD}{1!} + \frac{(LD)^2}{2!} + \cdots\right)$$

This is a good place to step away from the weeds to handwave

A condition on F that makes this work is to assume that

$$|F(t, y_a) - F(t, y_b)| \leq L\,|y_a - y_b|$$

for some fixed bound L which works for all points y_a, y_b. This condition says that F can change, but it cannot change too quickly.

(This is known as a Lipschitz condition.)

With this extra assumption, we can compute a better bound:

$$|y_{n+1} - y_n| \leq ML^n \frac{D^{n+1}}{(n+1)!}$$

Putting these together we get a bound for the telescoping series

$$(y_0 - y_1) + (y_1 - y_2) + (y_2 - y_3) + \cdots$$

Namely, its magnitude is bounded by:

$$y_0 + \frac{M}{L}\left(\frac{LD}{1!} + \frac{(LD)^2}{2!} + \cdots\right)$$

The series in the parentheses is the series for $e^{LD} - 1$, which we know converges to a finite number.

This is a good place to step away from the weeds to handwave

A condition on F that makes this work is to assume that

$$|F(t, y_a) - F(t, y_b)| \leq L\,|y_a - y_b|$$

for some fixed bound L which works for all points y_a, y_b. This condition says that F can change, but it cannot change too quickly.

(This is known as a Lipschitz condition.)

With this extra assumption, we can compute a better bound:

$$|y_{n+1} - y_n| \leq ML^n \frac{D^{n+1}}{(n+1)!}$$

Putting these together we get a bound for the telescoping series

$$(y_0 - y_1) + (y_1 - y_2) + (y_2 - y_3) + \cdots$$

Namely, its magnitude is bounded by:

$$y_0 + \frac{M}{L}\left(\frac{LD}{1!} + \frac{(LD)^2}{2!} + \cdots\right)$$

The series in the parentheses is the series for $e^{LD} - 1$, which we know converges to a finite number.

So the sequence $y_0, y_1, \ldots$ of functions converges uniformly, call the limit y.

A reminder of our goal

We want to check that the limit y of the Picard approximations satisfies the integral equation

$$y(x) = y_0 + \int_{x_0}^{x} F(t, y(t))\,\mathrm{d}t.$$

A reminder of our goal

We want to check that the limit y of the Picard approximations satisfies the integral equation

$$y(x) = y_0 + \int_{x_0}^{x} F(t, y(t))\,\mathrm{d}t.$$

Knowing that $y_n \to y$ uniformly, we can check:

$$\begin{aligned}
\lim_{n\to\infty} y_n(x) &= \lim_{n\to\infty} \left(y_0 + \int_{x_0}^{x} F(t, y_{n-1}(t))\,\mathrm{d}t \right) \\
&= y_0 + \int_{x_0}^{x} \lim_{n\to\infty} F(t, y_{n-1}(t))\,\mathrm{d}t \\
&= y_0 + \int_{x_0}^{x} F(t, \lim_{n\to\infty} y_{n-1}(t))\,\mathrm{d}t
\end{aligned}$$

Using the Lipschitz condition on F, we can say that this last step is valid, we can uniformly bring the limit inside F.

Summing things up

Theorem (Picard–Lindelöf theorem)

Consider the differential equation

$$y' = F(x, y),$$

where F satisfies a certain Lipschitz condition on a rectangle R, and consider a point (x_0, y_0) in R. Then there is a unique solution to this equation, valid on the a subdomain of the rectangle, satisfying the initial condition $y(x_0) = y_0$, where (x_0, y_0) is a point in R.

Summing things up

Theorem (Picard–Lindelöf theorem)

Consider the differential equation

$$y' = F(x, y),$$

where F satisfies a certain Lipschitz condition on a rectangle R, and consider a point (x_0, y_0) in R. Then there is a unique solution to this equation, valid on the a subdomain of the rectangle, satisfying the initial condition $y(x_0) = y_0$, where (x_0, y_0) is a point in R.

(You need a little more work than what we covered to check uniqueness.)

Summing things up

Theorem (Picard–Lindelöf theorem)

Consider the differential equation

$$y' = F(x, y),$$

where F satisfies a certain Lipschitz condition on a rectangle R, and consider a point (x_0, y_0) in R. Then there is a unique solution to this equation, valid on the a subdomain of the rectangle, satisfying the initial condition $y(x_0) = y_0$, where (x_0, y_0) is a point in R.

(You need a little more work than what we covered to check uniqueness.)

What about with n equations?

$$\begin{aligned} y_1' &= F_1(x, y_1, \ldots, y_n) \\ &\vdots \\ y_n' &= F_n(x, y_1, \ldots, y_n) \end{aligned}$$

Summing things up

Theorem (Picard–Lindelöf theorem)

Consider the differential equation

$$y' = F(x, y),$$

where F satisfies a certain Lipschitz condition on a rectangle R, and consider a point (x_0, y_0) in R. Then there is a unique solution to this equation, valid on the a subdomain of the rectangle, satisfying the initial condition $y(x_0) = y_0$, where (x_0, y_0) is a point in R.

(You need a little more work than what we covered to check uniqueness.)

What about with n equations?

$$\begin{aligned} y_1' &= F_1(x, y_1, \dots, y_n) \\ &\vdots \\ y_n' &= F_n(x, y_1, \dots, y_n) \end{aligned}$$

You can get a simultaneous solution to all n equations by a similar process, except doing it for all equations at once.

Summing things up

Theorem (Picard–Lindelöf theorem)

Consider the differential equation

$$y' = F(x, y),$$

where F satisfies a certain Lipschitz condition on a rectangle R, and consider a point (x_0, y_0) in R. Then there is a unique solution to this equation, valid on the a subdomain of the rectangle, satisfying the initial condition $y(x_0) = y_0$, where (x_0, y_0) is a point in R.

(You need a little more work than what we covered to check uniqueness.)

What about with n equations?

$$\begin{aligned} y_1' &= F_1(x, y_1, \ldots, y_n) \\ &\vdots \\ y_n' &= F_n(x, y_1, \ldots, y_n) \end{aligned}$$

You can get a simultaneous solution to all n equations by a similar process, except doing it for all equations at once.

So we get solutions to linear differential equations of order > 1.

Coda: Another existence theorem

We had to introduce this weird Lipschitz condition to make the calculations work out. Was this necessary?

Coda: Another existence theorem

We had to introduce this weird Lipschitz condition to make the calculations work out. Was this necessary?

Theorem (Peano existence theorem)

Consider the differential equation $y' = F(x, y)$ where F is continuous on a rectangle R, and consider a point (x_0, y_0) in R. Then there is a solution—not necessarily unique—to this equation, valid on a subdomain of the rectangle, satisfying $y(x_0) = y_0$.

Coda: Another existence theorem

We had to introduce this weird Lipschitz condition to make the calculations work out. Was this necessary?

Theorem (Peano existence theorem)

Consider the differential equation $y' = F(x, y)$ where F is continuous on a rectangle R, and consider a point (x_0, y_0) in R. Then there is a solution—not necessarily unique—to this equation, valid on a subdomain of the rectangle, satisfying $y(x_0) = y_0$.

That is, you can drop the assumption, but it comes at the cost of uniqueness.

Coda: Another existence theorem

We had to introduce this weird Lipschitz condition to make the calculations work out. Was this necessary?

Theorem (Peano existence theorem)

Consider the differential equation $y' = F(x, y)$ where F is continuous on a rectangle R, and consider a point (x_0, y_0) in R. Then there is a solution—not necessarily unique—to this equation, valid on a subdomain of the rectangle, satisfying $y(x_0) = y_0$.

That is, you can drop the assumption, but it comes at the cost of uniqueness.

For example, the equation

$$y' = \sqrt{|y|}$$

has multiple different solutions satisfying $y(0) = 0$:

$$y_1 = 0$$
$$y_2 = \frac{x^2}{4}$$