Math 302: Exact differential equations

Kameryn J Williams

University of Hawaiʻi at Mānoa

Spring 2021

A theorem from Calculus III

Theorem (Schwarz's theorem)

Suppose $f(x,y) : \mathbb{R}^2 \to \mathbb{R}$ *has continuous second partial derivatives. Then,*

$$\frac{\partial^2}{\partial x \partial y} f(x,y) = \frac{\partial^2}{\partial y \partial x} f(x,y)$$

A theorem from Calculus III

Theorem (Schwarz's theorem)

Suppose $f(x,y) : \mathbb{R}^2 \to \mathbb{R}$ has continuous second partial derivatives. Then,

$$\frac{\partial^2}{\partial x \partial y} f(x,y) = \frac{\partial^2}{\partial y \partial x} f(x,y)$$
$$f_{xy}(x,y) = f_{yx}(x,y)$$

A theorem from Calculus III

Theorem (Schwarz's theorem)

Suppose $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$ has continuous second partial derivatives. Then,

$$\frac{\partial^2}{\partial x \partial y} f(x, y) = \frac{\partial^2}{\partial y \partial x} f(x, y)$$
$$f_{xy}(x, y) = f_{yx}(x, y)$$

Let's look at what this theorem looks like from the perspective of the first partial derivatives.

- Suppose $P(x, y) = f_x(x, y)$ and $Q(x, y) = f_y(x, y)$.
- Then, $\frac{\partial}{\partial y} P(x, y) = \frac{\partial}{\partial x} Q(x, y)$.

A theorem from Calculus III

Theorem (Schwarz's theorem)

Suppose $f(x,y) : \mathbb{R}^2 \to \mathbb{R}$ *has continuous second partial derivatives. Then,*

$$\frac{\partial^2}{\partial x \partial y} f(x,y) = \frac{\partial^2}{\partial y \partial x} f(x,y)$$
$$f_{xy}(x,y) = f_{yx}(x,y)$$

Let's look at what this theorem looks like from the perspective of the first partial derivatives.

- Suppose $P(x,y) = f_x(x,y)$ and $Q(x,y) = f_y(x,y)$.
- Then, $\frac{\partial}{\partial y} P(x,y) = \frac{\partial}{\partial x} Q(x,y)$.
- And also $\displaystyle\int P(x,y)\,\mathrm{d}x = \int Q(x,y)\,\mathrm{d}y$.

A question

Question

Suppose you have two nice functions $P(x, y)$ and $Q(x, y)$ of two variables, and that $P_y = Q_x$. Must it be that

$$\int P(x, y)\,\mathrm{d}x = \int Q(x, y)\,\mathrm{d}y?$$

In other words, if $P_y = Q_x$, must it be that P and Q are the first partial derivatives of some function $f(x, y)$?

A question

Question

Suppose you have two nice functions $P(x, y)$ and $Q(x, y)$ of two variables, and that $P_y = Q_x$. Must it be that

$$\int P(x, y)\,\mathrm{d}x = \int Q(x, y)\,\mathrm{d}y?$$

In other words, if $P_y = Q_x$, must it be that P and Q are the first partial derivatives of some function $f(x, y)$?

The answer, somewhat amazingly, is yes. If our functions are nice then just satisfying $P_y = Q_x$ is enough to guarantee that they come from differentiating the same function with respect to different variables.

Exact differentials

Definition

Consider a differential $P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y$. It is called exact if there is a function $f(x,y)$ so that $P(x,y) = f_x(x,y)$ and $Q(x,y) = f_y(x,y)$.

Exact differentials

Definition

Consider a differential $P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y$. It is called exact if there is a function $f(x, y)$ so that $P(x, y) = f_x(x, y)$ and $Q(x, y) = f_y(x, y)$.

Theorem

The differential $P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y$ is exact if and only if there is a simply connected region in which P_y and Q_x are defined and continuous and $P_y = Q_x$.

Simply connected regions

In short, a region in the plane is simply connected if it has no holes. If you would like a more formal definition, take a topology class.

Finding the $f(x, y)$ for an exact differential

I want to sketch how to prove this theorem, as it will show how to find the desired $f(x, y)$.

Exact differential equations

A differential equation

$$P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y$$

is exact if the differential $P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y$ is exact.

- The solution to an exact differential equation is

$$f(x, y) = C$$

  where $P = f_x$ and $Q = f_y$.

An exact differential equation

$$(x^2 + y^2 + 1)y' + 2xy + x^2 + 3 = 0$$

The general method

Consider a differential equation

$$P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y.$$

What do you do if you think it might be exact?

The general method

Consider a differential equation

$$P(x, y)\,dx + Q(x, y)\,dy.$$

What do you do if you think it might be exact?

1. To confirm whether it is exact, compute P_y and Q_x and check whether they are equal. If not, you'll have to use another method.
2. If $P_y = Q_x$ then you want to find the $f(x, y)$ that you differentiate to get $P(x, y)$ and $Q(x, y)$.
3. Antidifferentiating $P(x, y)$ wrt x will tell you all terms in $f(x, y)$ which depend on x.
4. Antidifferentiating $Q(x, y)$ wrt y will tell you all terms in $f(x, y)$ which depend on y.
5. Putting those together will tell you all terms in $f(x, y)$ (except any constant term, but that gets absorbed into the C).
6. Be careful not to double-count terms involving both x and y!

Another example

$$(2x + ye^x + xye^x)\,\mathrm{d}x + xe^x\,\mathrm{d}y = 0$$

Another example

$$(2x + ye^x + xye^x)\,\mathrm{d}x + xe^x\,\mathrm{d}y = 0$$

Find the particular solution satisfying $y(1) = 1$.