Math 302: Complex numbers and linear independence of functions

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Higher-order linear differential equations

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Now we want to turn to higher-order differential equations, those involving second derivatives or higher.

- As you might expect, higher-order differential equations are generally more difficult to analyze than first-order differential equations.
- We will focus on the linear higher-order differential equations, namely those of the form

$$a_n(x)y^{(n)} + \cdots + a_1(x)y' + a_0(x)y = b(x).$$

A detour

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- Some facts about complex numbers; and
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- Some facts about complex numbers; and
- Some facts from linear algebra.

Depending on what math classes you've taken, you may have seen some of this already. But since it's not a required pre-requisite to take this class, I want to take some lecture time to go over this material.

Real and complex numbers

One reason the real numbers are useful because they allow us to do calculus.

- You get $\mathbb{R}$ from $\mathbb{Q}$ by "filling in the gaps", and this is what makes the machinery of limits, derivatives, and so on work.
- Many important theorems of calculus, such as the intermediate value theorem, aren't true if you restrict to $\mathbb{Q}$, and need these gaps filled in to be true.
  (For example, if you look only at rational numbers, the function $f(x) = x^2 - 2$ crosses from negative to positive without hitting 0.)

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But sometimes the real numbers aren't enough, and we want to extend to the complex numbers.

- Complex numbers ($\mathbb{C}$) are those of the form $a + bi$ where $a, b \in \mathbb{R}$ and i is a new number which is defined by the relation $i^2 = -1$.
- The idea: negative numbers don't have square roots in $\mathbb{R}$, so we expand $\mathbb{R}$ by adding square roots for negative numbers.

The fundamental theorem of algebra

Some polynomials, such as $x^2 + 1$, don't have roots in $\mathbb{R}$. But they do have complex roots.

Theorem (The fundamental theorem of algebra)

Any non-constant polynomial with complex coefficients has a root. That is, if you consider the polynomial, with coefficients $a_i \in \mathbb{C}$ and $n > 0$,

$$a_n z^n + \cdots a_1 z + a_0,$$

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We won't prove this theorem, but you can prove it using complex analysis—calculus with complex numbers and functions instead of real numbers and functions.

The complex plane

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You can also describe complex numbers using polar coordinates:

- The absolute value or modulus of $z = a + bi$ is $|z| = \sqrt{a^2 + b^2}$.
- The argument of $z = a + bi$ is $\arg z = \arctan(b/a)$.

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For reasons that will explained in a few slides, to write complex numbers in polar coordinates we write

$$z = re^{i\theta},$$

where $r = |z|$ and $\theta = \arg z$.

Algebra with complex numbers

You can add subtract and multiply complex numbers written in rectangular form by using the rules for algebra with binomials, and remembering that $i^2 = -1$:

- $(a + bi) + (c + di) = (a + c) + (b + d)i$
- $(a + bi) - (c + di) = (a - c) + (b - d)i$
- $(a + bi) \cdot (c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$

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Division is a little trickier, and needs a new idea. The problem is, how do we get rid of the i in the denominator?

Algebra with complex numbers

- The conjugate of $z = a + bi$ is $\bar{z} = a - bi$.
- Note that $z \cdot \bar{z} = (a + bi)(a - bi) = a^2 + b^2 = |z|^2$ is always real.

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$$\frac{a+bi}{c+di} = \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} = \frac{(a+bi)(c-di)}{c^2+d^2} = \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$$

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Actually doing this by hand is a bit tedious, and on the rare event that you have to compute with complex numbers for this class, I encourage you to use a calculator or computer.

A big interest here is theoretical: we've seen that $\mathbb{C}$ has all the same algebraic operations as $\mathbb{R}$. (But we don't have an order $<$ that's compatible with those algebraic operations.) So the same sorts of things you can do with real numbers you can also do with complex numbers.

Algebra with complex numbers

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Algebra with complex numbers

Polar form is not convenient for adding/substracting complex numbers.

But it does make multiplication/division straightforward, using rules for exponents:

- $re^{i\theta} \cdot se^{i\varphi} = rse^{i(\theta+\varphi)}$
- $\dfrac{re^{i\theta}}{se^{i\varphi}} = \dfrac{r}{s}e^{i(\theta-\varphi)}$

The complex exponential function

We can use the Taylor series for e^x to define e^z for a complex input z:

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \cdots$$

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You can show that this series converges for any complex number z, so it defines a function on the entire complex plane. This is done by exactly the same work as in the real case—you show that the radius of convergence is ∞, so it always converges.

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What does it look like if we input a purely imaginary number?

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&= 1 + ix - \frac{x^2}{2!} - i \cdot \frac{x^3}{3!} + \frac{x^4}{4!} + i \cdot \frac{x^5}{5!} - \frac{x^6}{6!} \pm \cdots
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&= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \pm \cdots\right) + \\
&\quad i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \pm \cdots\right)
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& \quad i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \pm \cdots\right) \\
&= \cos x + i \sin x
\end{aligned}
$$

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Using this equation and rules for exponents, we can rewrite e^{a+bi}:

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So the real part of z determines the modulus of e^z, and the imaginary part of z determines the modulus of e^z.

- This also justifies the polar form notation for complex numbers:

$$re^{i\theta} = e(\cos\theta + i\sin\theta)$$

Complex trigonometric functions

Just like we defined e^z using the Taylor series for the exponential function, we can do the same for $\sin z$ and $\cos z$:

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} \pm \cdots$$

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You can use these definitions to check how to write $\sin z$ and $\cos z$ in terms of the exponential function:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = \frac{1}{i}\sinh(iz)$$
$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = \cosh(iz)$$

Complex numbers

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To summarize:

- You can extend $\mathbb{R}$ to $\mathbb{C}$ by adding square roots for negative numbers, getting a 2d plane of complex numbers.
- $\mathbb{C}$ has the same algebraic operations as $\mathbb{R}$. (In math jargon, they are both fields.)
- You can extend the exponential and trig functions to $\mathbb{C}$, and doing so reveals that they are tightly connected.
- An important equation: $e^{a+bi} = e^a(\cos b + i \sin b)$.

Some linear algebra

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- A fundamental concept in linear algebra is a vector space.
- A vector space over $\mathbb{R}$ (over $\mathbb{C}$) consists of vectors which can be added and multiplied by scalars from $\mathbb{R}$ (from $\mathbb{C}$).

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Examples:

- $\mathbb{R}^n$ is a vector space over $\mathbb{R}$: vector addition and scalar multiplication are coordinate-wise.
- $\mathbb{C}$ is a vector space over $\mathbb{R}$: it is isomorphic ($=$ the same) to $\mathbb{R}^2$.

Vector spaces of functions

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$$(x^3 + 2x^2 + x + 3) + (3x^2 - 2x - 2) = x^3 + 5x^2 - x + 1$$
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- The collection of *it* all functions $f : \mathbb{R} \to \mathbb{R}$ form a vector space: $f + g$ is their sum as functions, and $c \cdot f$ is scalar multiplication with functions.
- Restricting to just continuous functions $f : \mathbb{R} \to \mathbb{R}$, or just differentiable $f : \mathbb{R} \to \mathbb{R}$, or just infinitely differentiable $f : \mathbb{R} \to \mathbb{R}$, or just integrable $f : \mathbb{R} \to \mathbb{R}$, or ..., all form vector spaces.

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- A linear combination of vectors is a sum of scalar multiples of those vectors. A linear combination is nontrivial if at least one scalar multiple is nonzero.
- A collection of vectors is linearly independent if you cannot write a vector in the collection as a nontrivial linear combinations of the others.
- It's often more useful to think in terms of linear dependence. If you have vectors $v_1, v_2, \ldots, v_k$, they are linearly dependent if and only if there are scalars $c_1, c_2, \ldots, c_k$, at least one nonzero, so that

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$$