Math 302: The catenary

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The catenary

Robert Hooke, English scientist and architect.

Hold a flexible chain, rope, cable, or similar at two points of equal height, and let it hang freely.

This curve is called the catenary (from the Latin word *catēna*, meaning chain).

Can we describe this curve?

Catenary arches

One application of this is in architecture:

- The catenary is the curve which makes it so that tension is entirely in the direction tangent to the curve.
- This makes it well-suited as a shape for arches: the arch supports its own weight well because the force is tangent to the curve of the arch.

Rainbow Bridge in Utah, a naturally-occuring arch which takes the shape of an inverted catenary.

The assumptions

- A chain is hung from two points of equal height.
- The chain is at rest, and the only forces on the chain are tension and gravity.
- The chain is thin, so it is accurately modeled as a 1d curve.
- The chain is uniform in density, so the weight of a segment depends only on its length,
- The chain is flexible, so any tension exerted on it is tangent to the curve.

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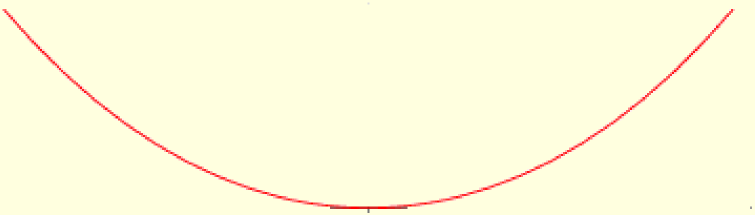
Of course, you can also ask what happens if you drop some of those assumptions. That complicates the analysis, and we will stick with this simplest setup.

Setting things up



- Parameterize the curve by $\vec{r}(s) = \langle x(s), y(s) \rangle$, where s is arc length.
- Pick the bottom of the curve to be the basepoint $s = 0$.
- Because we parameterized by arc length, $\frac{\mathrm{d}\vec{r}}{\mathrm{d}s}$ is always a unit vector.

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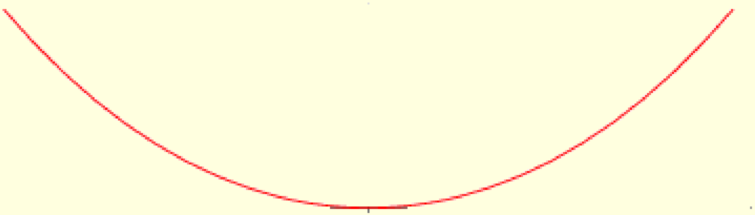

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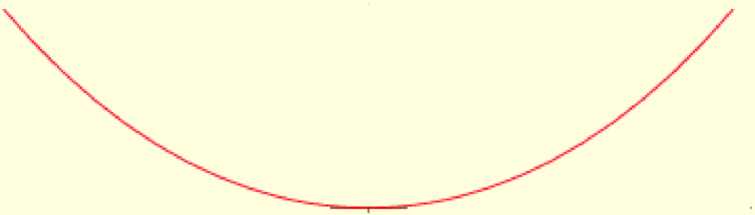

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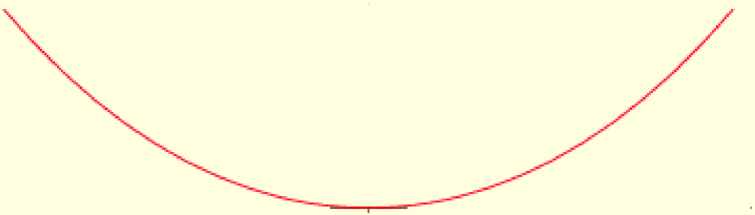
- Pick a point $\vec{r}$ on the right of the curve, i.e. $s > 0$. (This is enough to consider, since the left case is symmetric.)
- We want to analyze the forces acting on the segment of the chain from the basepoint to $\vec{r}$.

Analyzing the forces



Three forces on the segment from the basepoint to $\vec{r}$:

- The tension $\vec{T}_0$ at the basepoint;
- The tension $\vec{T}$ at $\vec{r}$; and
- The weight $\vec{W}$.

These are in equilibrium, so the three vectors sum to the zero vector.

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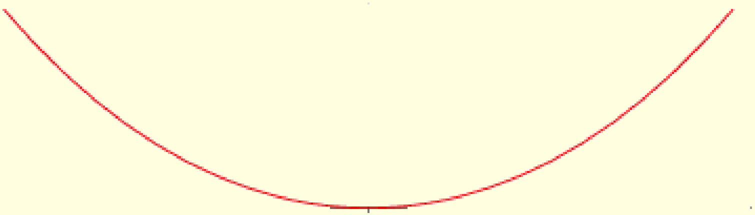

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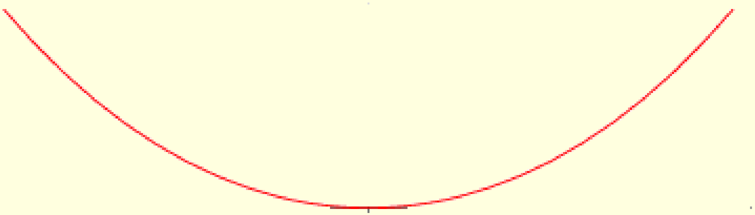

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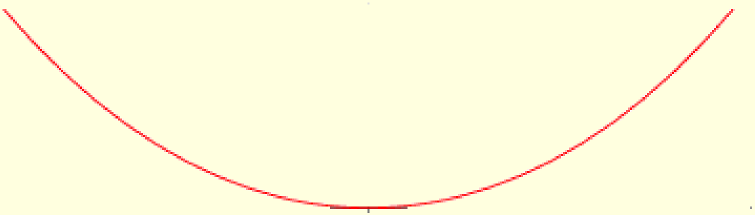

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Take a ratio:

$$\frac{T\sin\theta}{T\cos\theta} = \tan\theta = \frac{dy}{dx} = g\delta s/T_0$$

Combine the constants into one: $a = T_0/(g\delta)$.

$$\frac{dy}{dx} = \frac{s}{a}$$

A differential equation $\to$ better differential equations

We arrived at a differential equation which describes the slope of the curve:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{s}{a},$$

where a is a constant and s is the arc length from the bottom of the curve.

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Therefore:

$$\frac{dx}{ds} = \frac{a}{\sqrt{a^2 + s^2}}$$

and

$$\frac{dy}{ds} = \frac{dy}{dx} \cdot \frac{dx}{ds} = \frac{s}{\sqrt{a^2 + s^2}}.$$

Two differential equations to solve

$$\frac{\mathrm{d}y}{\mathrm{d}s} = \frac{s}{\sqrt{a^2 + s^2}} \qquad \frac{\mathrm{d}x}{\mathrm{d}s} = \frac{a}{\sqrt{a^2 + s^2}}$$

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$$\frac{dy}{ds} = \frac{s}{\sqrt{a^2 + s^2}}$$

This can be solved by integrating the righthand side, doing integration by substitution:

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This can also be solved by integrating. It's not so simple, so let's use a computer algebra system:

$$x(s) = a \log\left(\frac{s + \sqrt{a^2 + s^2}}{a}\right) + C$$

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This looks pretty ugly. Let's take a detour to investigate this function more closely

The arcsinh function

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Let's define a new function (the name will be explained later):

$$\operatorname{arcsinh}(u) = \log\left(u + \sqrt{1 + u^2}\right).$$

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Some observations:

- arcsinh is defined as a composition of strictly increasing functions:
    - The derivative of $u + \sqrt{1 + u^2}$ is $1 + \frac{u}{\sqrt{1+u^2}}$, which is always strictly positive.
    - So $u + \sqrt{1 + u^2}$ is strictly increasing.
    - And log is strictly increasing.
- So arcsinh is strictly increasing.

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  - So $u + \sqrt{1 + u^2}$ is strictly increasing.
  - And log is strictly increasing.
- So arcsinh is strictly increasing.
- Because arcsinh is strictly increasing, it is one-to-one, and so it has an inverse.
- What does this inverse look like?

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To explain the name:

- The circular trig functions sin and cos satisfy the equation $\cos^2 u + \sin^2 u = 1$
- This should make you think of the equation $x^2 + y^2 = 1$ which describes the unit circle.
- The hyperbolic trig functions sinh and cosh satisfy the equation $\cosh^2 u - \sinh^2 u = 1$.
- This should make you think of the equation $x^2 - y^2 = 1$ which describes a hyperbola.

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And this can in turn be used to describe y in terms of x.

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Therefore:

$$\begin{aligned} y &= \sqrt{a^2 + s^2} \\ &= \sqrt{a^2 + a^2\sinh^2(x/a)} \\ &= \sqrt{a^2\cosh^2(x/a)} \\ &= a\cosh(x/a) \end{aligned}$$

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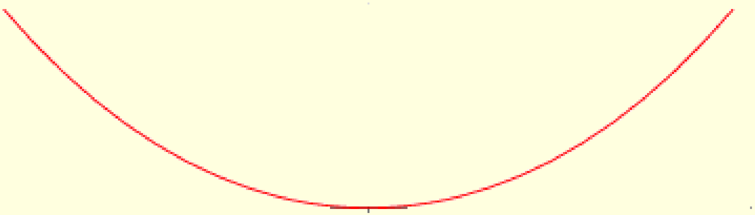
So we've found a way to describe the catenary where y is a function of x.

Conclusion



The curve can be described parametrically: $\vec{r}(s) = \langle x(s), y(s) \rangle$, where

$$x(s) = a \operatorname{arcsinh}(s/a)$$
$$y(s) = \sqrt{a^2 + s^2}$$

We can also describe it by expressing y as a function of x:

$$y = a\cosh(x/a) = a \cdot \frac{e^{x/a} + e^{-x/a}}{2}$$

The constant $a = T_0/(g\delta)$ is based only upon the acceleration g due to gravity, the density δ of the chain, and the tension T_0 at the bottom point of the chain.