Math 302: Linear ODEs and the Bernoulli equation

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First-order linear differential equations

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- If you've taken linear algebra: an equation like $Ax + By = C$ is a linear equation in x and y. Linear differential equations are linear equations in y and y', but where the coefficients are functions of x rather than real numbers.
- If you have something of the form $A(x)y' + B(x)y = C(x)$ you can get it in the standard form by moving to $y' + \frac{B(x)}{A(x)}y = \frac{C(x)}{A(x)}$.

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- Warning! An ODE being linear is different from it having linear coefficients, like we talked about two weeks ago.

An example

$$\begin{aligned} y' + \frac{y}{x} &= x^2 \\ \mathrm{d}y + (y/x - x^2)\,\mathrm{d}x &= 0 \end{aligned}$$

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$$y' + \frac{y}{x} = x^2$$
$$dy + (y/x - x^2)\,dx = 0$$

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So where did the integrating factor x come from?

The general solution

Theorem

$m(x) = \exp(\int P(x)\,\mathrm{d}x)$ is an integrating factor for the linear differential equation

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Theorem

$m(x) = \exp(\int P(x)\,\mathrm{d}x)$ *is an integrating factor for the linear differential equation*

$$y' + P(x)y = Q(x).$$

This is actually not so bad to verify.

The general solution

Given a linear differential equation $y' + P(x)y = Q(x)$ we multiply by an integrating factor and rearrange to get the exact differential equation

$$\exp\left(\int P(x)\,\mathrm{d}x\right)\,\mathrm{d}y + \exp\left(\int P(x)\,\mathrm{d}x\right)(P(x)y - Q(x))\,\mathrm{d}x = 0.$$

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We can solve this by the methods we learned last week, but there's another method using substitution:

$$u = y\exp\left(\int P(x)\,\mathrm{d}x\right).$$

The general solution

Given a linear differential equation $y' + P(x)y = Q(x)$ we multiply by an integrating factor, rearrange, and then substitute to get

$$\mathrm{d}u = \exp\left(\int P(x)\,\mathrm{d}x\right) Q(x)\mathrm{d}x,$$

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$$y = \frac{\int \exp\left(\int P(x)\,\mathrm{d}x\right) Q(x)\,\mathrm{d}x + C}{\exp\left(\int P(x)\,\mathrm{d}x\right)}$$

An example

$$\frac{\mathrm{d}x}{\mathrm{d}y} + 2yx = e^{-y^2}$$

Another example

L, R, E, and k are constants, I and t are variables.

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(Electrical engineers might recognize this as the equation describing a simple electrical circuit with an inductor, resistor, and an applied force.)

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If $n \neq 1$, we need a different technique. It's not linear, but we can make it linear with a substitution.

Solving Bernoulli equations

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$$y' + P(x)y = Q(x)y^n \qquad\qquad (n \neq 1)$$

Multiply both sides by $(1-n)y^{-n}$, then use the substitution

$$u = y^{1-n}, \qquad \mathrm{d}u = (1-n)y^{-n}\mathrm{d}y.$$

An example

$$y' + xy = \frac{x}{y^3}$$

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Find the particular solution with initial condition $y(0) = 2$.