Math 1332: Voting Theory, part II

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Last time

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	Alice	Barbara	Carlos	David	Eric	Fred	Grace	Holly	Ivan
1st choice	B	F	F	S	B	F	F	B	S
2nd choice	S	B	B	B	S	S	B	S	B
3rd choice	F	S	S	F	F	B	S	F	F

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- The plurality method may fail to select the Condorcet winner.
- But the Condorcet method may fail badly to select a choice, since there may be no Condorcet winner.
- The last method we looked, instant runoff voting simulates a multi-round runoff election with a single ballot.
- It seemed like it avoided the issues with the previous mmethods. But maybe it too has problems...

A problem with IRV

Consider the following vote to read one of Frege, Quine, or Russell, in a larger philosophy club with 30 members:

# of votes	11	7	3	9
1st choice	F	Q	Q	R
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IRV violates the monotonicity criterion: Voters changing their vote to rank an option more highly cannot make that option switch from winning to losing the vote.

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Voters make a full preference ballot as their vote, listing all options in order. Points are then assigned to each option based on its ranking: 1 for last place, 2 for second to last, and so on up to the maximum points being awarded for a 1st choice. The winner is the option with the most points.

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Yet another philosophy club is voting whom to read, this time from Maddy, Linnebo, and Shapiro.

# of votes	12	8
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Majority criterion: If an option has the majority of 1st choice votes, then it should win.

Revisiting the Condorcet criterion

We've been focusing so much on other criteria that we've mostly forgotten about the Condorcet criterion.

- If a choice wins in all one-to-one match-ups, then it should win the vote.

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Voters make a full preference ballot as their vote, listing all options in order. Points are assigned by looking at one-to-one match-ups accumlated across all ballots. A win is worth 1 point for the winner, while in a tie they split it for $1/2$ point each.

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This is the most complicated method we've looked at yet, so let's see an example.

Copeland's method

A philosophy club is having a vote to decide which phenomenologist to read, Husserl, Heidegger, or Merleau-Ponty.

	Alice	Barbara	Carlos	David	Eric	Fred
1st choice	Hu	Me	Me	Hu	He	He
2nd choice	Me	Hu	Hu	He	Hu	Me
3rd choice	He	He	He	Me	Me	Hu

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(Before we calculate the Copeland's method winner, let's note that this is a 2 vs 2 vs 2 three-way tie, so both the plurality method and IRV method will flounder to resolve the tie.)

We look at the pairwise matchups:

He vs Hu	2	4
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We then assign points based on how many matchups are won, splitting the points on a tie:

Heidegger	$\frac{1}{2}$
Husserl	$1 + \frac{1}{2}$
Merleau-Ponty	$\frac{1}{2}$

A problem with Copeland's method

Let's revisit the last vote:

We computed the one-versus-one matchups:

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What if Heidegger wasn't an option at all?

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Removing the third option made it a tie!

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Another fairness criterion

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Here's a joke which illustrates the criterion:

- The American philosopher Morgenbesser was having dinner at a New York diner. When ordering dessert, the waiter tells him that the options are apple pie and blueberry pie. He orders apple. A couple minutes later the waiter comes back and tells him that there's also cherry pie. Morgenbesser responds, "In that case, I'll have the blueberry."

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- The majority criterion: If a choice gets the majority of 1st choice votes, it wins.
- The IIA criterion: Whether a choice A is preferred to a choice B doesn't depend on the existence of alternatives $C, D, \ldots$
- The Condorcet criterion: If a choice is preferred over all others in one-versus-one comparisons, it should win.
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- Copeland's method satisfies the majority, monotonicity, and Condorcet criteria, but not the IIA criterion.
- The Borda count method satisfies the monotonicity, IIA, and Condorcet criteria, but not the majority criterion.
- The IRV method satisfies the majority and IIA criteria but not the Condorcet or monotonicity criteria.
- The plurality voting method satisfies the majority, monotonicity, and IIA criteria, but not the Condorcet criterion.

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Can a voting method satisfy all these fairness criteria simultaneously?

Arrow's impossibility theorem

Theorem (Arrow's impossibility theorem)

No voting method applied to choosing among 3 or more choices can satisfy all of these fairness criteria:

- *The majority criterion;*
- *The independence of irrelevant alternatives criterion;*
- *The Condorcet criterion; and*
- *The monotonicity criterion.*

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- Mathematizing the voting process is what enables Arrow's theorem to be stated and proved.
- So even though the mathematical look ignores important real-world context, it still allows us to draw useful conclusions.
- There's no perfectly fair voting method, if we take a restrictive view that only looks at what can be mathematized.
- So when it comes to designing voting methods for real-world use, we can't just blindly pick a unique best option. Instead, we have to weigh different pros and cons.