MATH455 HOMEWORK 3
DUE FRIDAY, FEBRUARY 14

Exercise 1. Let $\mathcal{M}$, $\mathcal{N}$, and $\mathcal{K}$ be structures in the same language. Show that if $M \prec K$, $N \prec K$, and $M \subseteq N$ then $M \prec N$.

Let $\mathcal{R} = (\mathbb{R}, 0, 1, +, \cdot, <)$ be the reals as an ordered field. Recall that a real r is *algebraic* if there is a polynomial $p(x)$ with integer coefficients so that $p(r) = 0$.

Exercise 2. Show that every algebraic real is definable without parameters in $\mathcal{R}$.[1]

Recall the definition of the functions on an ultraproduct $\mathcal{M} = \prod_{i \in I} \mathcal{M}_i / U$ using an ultrafilter U on I, where f is an arity k function symbol in the language of $\mathcal{M}$:

$$f^{\mathcal{M}}\big((x_i^1), \ldots, (x_i^k)\big) = (y_i) \qquad \text{iff} \qquad \big\{i \in I : f^{\mathcal{M}_i}(x_i^1, \ldots, x_i^k) = y_i\big\} \in U.$$

Exercise 3. Show that $f^{\mathcal{M}}$ is well-defined on the ultraproduct $\mathcal{M}$. That is, show that if $(x_i^\ell) =_U (\bar{x}_i^\ell)$ for all $\ell \leq k$ then $f^{\mathcal{M}}((x_i^1), \ldots, (x_i^k)) =_U f^{\mathcal{M}}((\bar{x}_i^1), \ldots, (\bar{x}_i^k))$. [Hint: this is similar to what we did in class for showing that $=_U$ is an equivalence relation and that $R^{\mathcal{M}}$ is a congruence modulo $=_U$.]

Exercise 4. Let U_p be the principal ultrafilter on I generated by $p \in I$, that is $A \subseteq I$ is in U_p iff $p \in A$. Show that $\prod_{i \in I} \mathcal{M}_i / U_p \cong \mathcal{M}_p$. (In particular, this implies that $\mathrm{Ult}(\mathcal{M}, U_p) \cong \mathcal{M}$.)

[1] In fact, the only definable elements in $\mathcal{R}$ are the algebraic reals, and if $\mathcal{A} \subseteq \mathcal{R}$ is the restriction of $\mathcal{R}$ to the algebraic reals then $\mathcal{A} \prec \mathcal{R}$. But this is much harder to prove.