MATH455 HOMEWORK 11
DUE FRIDAY, MAY 1

Throughout, $\mathcal{M}$ and $\mathcal{K}$ are models of arithmetic, i.e. structures satisfying Peano arithmetic, and $\mathcal{N} = \langle \mathbb{N}, 0, 1, +, \cdot, < \rangle$ is the standard model of arithmetic.

Say that $\mathcal{M}$ *embeds as an initial segment* of $\mathcal{K}$ if there is an embedding[1] $e : \mathcal{M} \to \mathcal{K}$ so that for all $k \in K$ if $k < m$ for some $m \in \operatorname{ran} e$ then $k \in \operatorname{ran} e$. If this embedding is the identity—i.e. if $\mathcal{M} \subseteq \mathcal{K}$—then we say that $\mathcal{M}$ is an *initial segment* of $\mathcal{K}$.

Exercise 1. Let $\mathcal{M} \models \mathsf{PA}$ be any structure satisfying the axioms of Peano arithmetic. Show that $\mathcal{N}$ embeds as an initial segment of $\mathcal{M}$. [Hint: Construct the embedding e by induction on $\mathbb{N}$.]

Say that $\mathcal{M} \prec_{\Delta_0} \mathcal{K}$, in words: $\mathcal{M}$ is a Δ_0-*elementary substructure* of $\mathcal{K}$, if $\mathcal{M} \subseteq \mathcal{K}$ and for all Δ_0 formulae $\varphi(\bar{x})$ and all $\bar{a} \in M$ we have $\mathcal{M} \models \varphi[\bar{a}]$ iff $\mathcal{K} \models \varphi[\bar{a}]$. This is like the notion of elementary substructure except we only ask for agreement on Δ_0 formulae.

Exercise 2. Suppose that $\mathcal{M}$ is an initial segment of $\mathcal{K}$. Show that $\mathcal{M} \prec_{\Delta_0} \mathcal{K}$. Conclude that if φ is a Σ_1 sentence so that $\mathcal{N} \models \varphi$ then $\mathsf{PA} \vdash \varphi$.

In lecture we saw that there are true Π_1 sentences which are independent of PA. The previous exercise shows that we cannot get this for true Σ_1 sentences; if φ is a Σ_1 sentence which is true in the standard model then PA proves φ.

[1] I.e. injective homomorphism