MATH455 HOMEWORK 1
DUE FRIDAY, JANUARY 31

Recall the three axioms for equivalence relations, in the language $\mathcal{L}$ with a binary relation symbol $\equiv$.

(Refl) $$\forall x\ x \equiv x$$

(Sym) $$\forall x \forall y\ x \equiv y \Rightarrow y \equiv x$$

(Trans) $$\forall x \forall y \forall z\ (x \equiv y \wedge y \equiv z) \Rightarrow x \equiv z$$

Exercise 1. Consider the following step-by-step proof of Refl from Sym + Trans as axioms.

(1) Fix x.
(2) Fix y so that $x \equiv y$.
(3) Then by Sym, $y \equiv x$.
(4) So by Trans, $x \equiv x$.
(5) Since x was arbitrary we have proved $\forall x\ x \equiv x$.

This proof has a gap. Identify in which step the gap appears. Formulate an extra axiom φ which fills the gap and write a proof of Refl from the axioms Sym + Trans + φ. Produce an $\mathcal{L}$-structure which satisfies Sym and Trans but not Refl.

Exercise 2. Do Exercise 4.9 from the textbook (page 38).

Exercise 3. Let φ and ψ be arbitrary formulae. Prove that the following pairs of formulae are logically equivalent.

- $\neg(\varphi \wedge \psi)$ and $\neg\varphi \vee \neg\psi$.
- $\neg(\varphi \vee \psi)$ and $\neg\varphi \wedge \neg\psi$.
- φ and $\neg\neg\varphi$.
- $\forall x \varphi$ and $\neg\exists x \neg\varphi$.
- $\exists x \varphi$ and $\neg\forall x \neg\varphi$.

Exercise 4. Use the previous exercise to show the following.

- Every formula is logically equivalent to a formula using only $\wedge$, $\neg$, and $\forall$ (i.e. no $\vee$ nor $\exists$).
- Every formula is logically equivalent to a formula using only $\vee$, $\neg$, and $\exists$ (i.e. no $\wedge$ nor $\forall$).

[Hint: argue by induction on formulae.]