Math 321: Relations, part IV: More Equivalence Relations

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Fall 2020

Previously in Math 321

Last time we talked about equivalence relations.

- These give an abstract notion of sameness.
- Formally: an equivalence relation on a set X is a reflexive, symmetric, transitive relation on X.
- An alternate perspective: an equivalence relation partions X into equivalence classes.

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For today I want to talk about one important use of equivalence relations in mathematics, namely to build new structures.

An example: Modular arithmetic

Let's consider $n = 4$ and the equivalence relation $\equiv \mod 4$, which let's call $\sim$ for short.

Another example: Building $\mathbb{Q}$ from $\mathbb{Z}$

Define $\sim$ on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ as $(a, b) \sim (c, d)$ iff $ad = bc$.

Yet another example: Building $\mathbb{Z}$ from $\mathbb{Q}$

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Define $\sim$ on $\mathbb{N} \times \mathbb{N}$ as $(a, b) \sim (c, d)$ iff $a + d = b + c$.

A vague sketch of the general picture

Suppose you have a mathematical structure X and $\sim$ is an equivalence relation which plays nicely with the structure of X. Then you can quotient X to produce a new structure on $X/\sim$, the collection of equivalence classes on X.

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This is a very vague sketch!

A final—incomplete—example: building $\mathbb{R}$ from $\mathbb{Q}$

Define $\sim$ on the set of infinite sequences of rational numbers as:

$$(p_n) \sim (q_n) \quad \text{iff} \quad \lim_{n\to\infty} |p_n - q_n| = 0$$

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We actually need to restrict this to the set of infinite sequences (p_n) so that

$$\lim_{n\to\infty} p_n - p_{n+1} = 0$$