# Math 321: Relations, part III: Equivalence Relations

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Fall 2020

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Today we're talking about another important kind of relation, nameley equivalence relations.

# Intuitive picture

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Example:

- The expressions $\frac{1}{2}$ and $\frac{2}{4}$ refer to the same rational number, even though they are different.

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Equivalence relations are a formalization of this concept.

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And that's it!

# Formal definition

A relation $\sim$ on a nonempty set $X$ is an equivalence relation if $\sim$ is reflexive, symmetric, and transitive.

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- Congruence of triangles, on the set of triangles in the plane.

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- Notice $[x]_\sim$ is never empty, because $x \in [x]_\sim$ (because $x \sim x$).

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- Notice $[x]_\sim$ is never empty, because $x \in [x]_\sim$ (because $x \sim x$).
- Then the equivalence classes partition $X$ into disjoint pieces.

# Equivalence classes are disjoint

## Proposition

*Let $\sim$ be an equivalence relation on $X$, and let $x, y \in X$. Then exactly one of the two cases holds.*

1. $[x] = [y]$, *which happens just in case* $x \sim y$; *or*
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Case 1 ($x \sim y$): Suppose $z \in [x]$. That is, $z \sim x$. Then, by transitivity we get that $z \sim y$, so $z \in [y]$. That is, we have seen $[x] \subseteq [y]$. Showing the other inclusion works similarly.

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Case 2 ($x \not\sim y$): Suppose toward a contradiction that $z \in [x] \cap [y]$. That is, $z \sim x$ and $z \sim y$. By symmetry we get $x \sim z$ and so by transitivity we get $x \sim y$, a contradiction. □

# Partitions

Indeed, we could define partitions first and from that define equivalence relations.

- Let $X$ be a nonempty set. A partition of $X$ is a collection $\mathcal{A}$ of nonempty subsets of $X$ so that:
  1. $\bigcup_{A \in \mathcal{A}} A = X$; and
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Let's make some observations:

- If $\mathcal{A}$ is a partition of $X$ then each $x \in X$ is in exactly one set in $\mathcal{A}$.
- If $\mathcal{A}$ is a partion of $X$ and $A, B \in \mathcal{A}$ then $A \cap B \neq \emptyset$ iff $A = B$.

# From partitions to equivalence relations

## Proposition

*Suppose $\mathcal{A}$ is a partition of nonempty $X$. Define a relation $\sim$ on $X$ as: $x \sim y$ iff $x$ and $y$ are in the same set in $\mathcal{A}$. Then $\sim$ is an equivalence relation.*

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($\sim$ is symmetric) If $x$ is in the same piece as $y$ then $y$ is in the same piece as $x$.
($\sim$ is transitive) Suppose $x \sim y$ and $y \sim z$. This is witnessed by sets $A, B \in \mathcal{A}$, i.e. $x, y \in A$ and $y, z \in B$. So $A \cap B \neq \emptyset$, which means that $A = B$. So we have seen that $x$ and $z$ are in the same piece of the partition. □

# Additional examples

For points $p, q \in \mathbb{R}^2$, let $d(p, q)$ be the Euclidean distance from $p$ to $q$. Set $p \sim q$ iff $d(p, 0) = d(q, 0)$.

# Additional examples

For sets $A, B \subseteq \mathbb{N}$, set $A \sim B$ iff the symmetric difference $A \triangle B = (A \setminus B) \cup (B \setminus A)$ is finite.

# Additional examples

For partitions $\mathcal{A}, \mathcal{B}$ of a finite set $X$, set $\mathcal{A} \sim \mathcal{B}$ iff $\mathcal{A}$ and $\mathcal{B}$ have the same number of pieces.