Math 321: Relations, part II: Orders

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Last time

We talked about how to make relations—such as $=$, $<$, $|$, or so on—mathematical objects. Specifically, in the most general form, a (binary) relation from a set A to a set B is a subset of the Cartesian product $A \times B$. That is, a relation is a set of ordered pairs (a, b) with $a \in A$ and $b \in B$.

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Today we're going to focus on a particularly important kind of relation, namely order relations.

Intuitive picture

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Some examples:

- $\leq$ on $\mathbb{N}$ compares numbers by size.
- You can think of $\leq$ on $\mathbb{R}$ as ordering the real line by position.
- You can think of $\subseteq$ as comparing sets by size.
- $\mid$ on $\mathbb{N}$ measures how complex a natural number is by how many primes you have to multiply together to get it.

Order relations

Recall these properties we talked about last time, for a relation R on a set A:

- R is reflexive if $a \, R \, a$ for all $a \in A$.
- R is transitive if $a \, R \, b$ and $b \, R \, c$ implies $a \, R \, c$ for all $a, b, c \in A$.

Let's introduce a couple new properties:

- R is antireflexive if $a \not R \, a$ for all $a \in A$.
- R is antisymmetric if $a \, R \, b$ and $a \neq b$ implies $b \not R \, a$ for all $a, b \in A$.
  - If R is reflexive, you can equivalently state antisymmetry as: $a \, R \, b$ and $b \, R \, a$ implies $a = b$ for all $a, b \in A$.
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  - If R is antireflexive, you can equivalently state antisymmetry as: $a \, R \, b$ implies $b \not R a$ for all $a, b \in A$.

If R is reflexive, transitive, and antisymmetric we call R a (nonstrict) order relation or (reflexive) order relation.

If R is antireflexive, transitive, and antisymmetric we call R a (strict) order relation or (antireflexive) order relation.

Examples

Let's look at $<$ on $\mathbb{Z}$, $\mid$ on $\mathbb{N}$, and $\subseteq$ on sets.

Strict versus nonstrict orders

Theorem

There is a correspondence between strict and nonstrict orders.

- *If* $\sqsubset$ *is a strict order on* A *then* $\sqsubseteq = \{(a, b) \in A : a \sqsubset b \text{ or } a = b\}$ *is a nonstrict order on* A.
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Homework :) $\square$

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Notation:

- A line underneath, as in $\subseteq$ or $\leq$ or $\sqsubseteq$, denotes a nonstrict order.
- A line underneath with a slash through it, as in $\subsetneq$ or $\lneq$, denotes a strict order. You may also see $\subsetneqq$ or $\lneqq$.
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    - Some authors—very rudely in my opinion–use $\subset$ to mean $\subseteq$. So to avoid confusion it's best to use $\subsetneq$ for the strict order.
- To denote the reverse of an order you flip the symbol horizontally: $\leq$ becomes $\geq$, $\subsetneq$ becomes $\supsetneq$, $\sqsubseteq$ becomes $\sqsupseteq$, and so on.

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Warning! $\nsubseteq$ does not have the same meaning as $\subsetneq$!

Partial and total orders

In some orders, everything is comparable.

- If x and y are real numbers, then $x < y$, $x = y$, or $x > y$.

In other orders, this is not the case.

- $A = \{0\}$ and $B = \{1\}$ are sets, but $A \not\subseteq B$ and $B \not\subseteq A$.
- 7 and 13 are natural numbers, but $7 \nmid 13$ and $13 \nmid 7$.

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- 7 and 13 are natural numbers, but $7 \nmid 13$ and $13 \nmid 7$.

We want to be able to distinguish these circumstances.

Partial and total orders

A set X equipped with an order $\leq$ on X is called a partially ordered set (or poset). We also call $\leq$ a partial order.

If $\leq$ satisfies the additional property that $x \leq y$ or $y \leq x$ for all $x, y \in X$, then we call it totally ordered or linearly ordered.

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Warning! Partial order does not mean non-total order. Rather, a total order is a special case of a partial order.
(Is this bad terminology? Probably.)

Extreme elements

Suppose $(X, \leq)$ is a poset.

- $m \in X$ is the smallest or least element if $m \leq x$ for all $x \in X$.
- $m \in X$ is a minimal element if there is no $x \in X$ so that $x < m$.

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Remark: If m is the smallest element in $(X, \leq)$ then m is minimal, and if M is the largest element then M is maximal. In both cases, it is the unique minimal/maximal element.

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Remark: A smallest element in $(X, \leq)$ is a largest element in $(X, \geq)$, and vice versa. A similar fact holds for minimal/maximal elements.

Extreme elements

Let's look at $<$ on $\mathbb{N}$, $\subseteq$ on $\mathcal{P}(\mathbb{N})$, and $\mid$ on $\{n \in \mathbb{N} : n \geq 2\}$.

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Proposition

Smallest elements, if they exist, are unique. The same is true for largest elements.

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Proof.

Suppose m and m' both satisfy the definition of being the smallest element of a partially ordered set $(X, \leq)$. Then, by definition, $m \leq m'$ and $m' \leq m$. So by antisymmetry $m = m'$.

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You could make a similar argument for largest elements, but I'd rather be lazy. As remarked before, if $(X, \leq)$ is a poset then so is $(X, \geq)$. So if $(X, \geq)$ has a smallest element it must be unique, but being a smallest element of $(X, \geq)$ is the same as being a largest element of $(X, \leq)$. □

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Proposition

In a total order, being a minimal element is equivalent to being a smallest element, and being a maximal element is equivalent to being a largest element.

For this reason, in total orders smallest elements are sometimes called minimums and largest elements are called maximums.

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Proof.

Like before we only have to check the minimal/smallest case.
($\Leftarrow$) We said earlier that smallest elements are minimal.
($\Rightarrow$) Suppose m is a minimal element of $(X, \leq)$. Consider any $x \in X$. By totalness, either $m < x$, $m = x$, or $m > x$. The last is impossible, because m is minimal, so $m \leq x$. So we have seen m is the smallest element of X. □

Upper bounds and lower bounds, suprema and infima

Let $(X, \leq)$ be a poset, and suppose $A \subseteq X$ is nonempty.

- $u \in X$ is an upper bound for A if $a \leq u$ for all $a \in A$.
- $\ell \in X$ is a lower bound for A if $a \geq \ell$ for all $a \in A$.

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- $\ell \in X$ is a lower bound for A if $a \geq \ell$ for all $a \in A$.
- Let U be the set of upper bounds for A. The smallest element of U, if it exists, is called the supremum or least upper bound for A.
- Let L be the set of lower bounds for A. The largest element of L, if it exists, is called the infimum or greatest lower bound for A.

Examples

Let's look at $\leq$ on $\mathbb{Q}$ and $\mathbb{R}$ and $\subseteq$ on $\mathcal{P}(\mathbb{N})$.