Math 321: Relations, part I

Kameryn J Williams

University of Hawaiʻi at Mānoa

Fall 2020

Mathematical objects

When we talk about mathematical objects, we don't just want to talk about them by themselves. There's nothing interesting in asserting: "the number 7". Rather, we want to say stuff like "$2 + 2 = 4$" or "$e < \pi$". That is, we want to apply functions like $+$ to our objects or we want to say our objects satisfy a relation like $<$.

Mathematical objects

When we talk about mathematical objects, we don't just want to talk about them by themselves. There's nothing interesting in asserting: "the number 7". Rather, we want to say stuff like "$2 + 2 = 4$" or "$e < \pi$". That is, we want to apply functions like $+$ to our objects or we want to say our objects satisfy a relation like $<$.

While something like "$2 + 2 < 7$ is a *statement* about mathematical objects, we also want to be able to think about the relation $<$ as a mathematical object in its own right.

Mathematical objects

When we talk about mathematical objects, we don't just want to talk about them by themselves. There's nothing interesting in asserting: "the number 7". Rather, we want to say stuff like "$2 + 2 = 4$" or "$e < \pi$". That is, we want to apply functions like $+$ to our objects or we want to say our objects satisfy a relation like $<$.

While something like "$2 + 2 < 7$ is a *statement* about mathematical objects, we also want to be able to think about the relation $<$ as a mathematical object in its own right.

So let's talk about how to do that.

Cartesian products

Definition

An ordered pair is just as the name suggests: a pair of objects where you know the order—which is first versus which is last. We write (a, b) for the ordered pair whose first element is a and whose second element is b.

Cartesian products

Definition

An ordered pair is just as the name suggests: a pair of objects where you know the order—which is first versus which is last. We write (a, b) for the ordered pair whose first element is a and whose second element is b.

Definition

Given two sets A and B, their Cartesian product is the set

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

Cartesian products

Definition

An ordered pair is just as the name suggests: a pair of objects where you know the order—which is first versus which is last. We write (a, b) for the ordered pair whose first element is a and whose second element is b.

Definition

Given two sets A and B, their Cartesian product is the set

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

For example, $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) : x, y \in \mathbb{R}\}$ is the Cartesian plane.

Properties of Cartesian products

1. If A has m elements and B has n elements then $A \times B$ has mn elements.
2. In general, $A \times B \neq B \times A$.
3. $A \times (B \cup C) = (A \times B) \cup (A \times C)$.
4. $A \times (B \cap C) = (A \times B) \cap (A \times C)$.
5. $A \times \emptyset = \emptyset \times A = \emptyset$.

Beyond two

You can also do Cartesian products with more than two coordinates, for example $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ is three-dimensional Euclidean space. If you have, say, four coordinates, then instead of ordered pairs (a, b) you need ordered quadruples (a, b, c, d). But the idea is the same, and we will mainly be concerned with the binary case.

What are relations?

Example: Consider the relation $<$ on $\mathbb{R}$. We want to think of this relation as an object.

What are relations?

Example: Consider the relation $<$ on $\mathbb{R}$. We want to think of this relation as an object.

- The idea: what we need to know is when $x < y$ is true.
- So we can represent $<$ as the set of all pairs (x, y) for which $x < y$.

What are relations?

Example: Consider the relation $<$ on $\mathbb{R}$. We want to think of this relation as an object.

- The idea: what we need to know is when $x < y$ is true.
- So we can represent $<$ as the set of all pairs (x, y) for which $x < y$.
- That is, we represent $<$ as a certain subset of the Cartesian product $\mathbb{R} \times \mathbb{R}$.

What are relations?

Example: Consider the relation $<$ on $\mathbb{R}$. We want to think of this relation as an object.

- The idea: what we need to know is when $x < y$ is true.
- So we can represent $<$ as the set of all pairs (x, y) for which $x < y$.
- That is, we represent $<$ as a certain subset of the Cartesian product $\mathbb{R} \times \mathbb{R}$.
- This perspective on relations is extensional—based only on what elements make the relation true—rather than intensional—based just on how it is defined.

Relations in general

Definition

Let A and B be sets. Then a (binary) relation from A to B is a subset of $A \times B$.

Relations in general

Definition

Let A and B be sets. Then a (binary) relation from A to B is a subset of $A \times B$.

Examples:

- $| = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a \text{ divides } b\}$.
- $< = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x < y\}$.
- Equivalence modulo n is the relation $\{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a \equiv b \mod n\}$.

Relations in general

Definition

Let A and B be sets. Then a (binary) relation from A to B is a subset of $A \times B$.

Examples:

- $| = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a \text{ divides } b\}$.
- $< = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x < y\}$.
- Equivalence modulo n is the relation $\{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a \equiv b \mod n\}$.

When talking about relations abstractly, we will need to give them a name. We will usually use a letter, saving symbols like $<$, $\subseteq$, $\in$, $|$, etc. for specific relations.

So $a\,R\,b$ just means that $(a, b) \in R$ for some relation R.

Thinking pictorally about relations

Properties of relations

Let R be a relation from A to B.

- The domain of R is $\operatorname{dom} R = \{a \in A : \exists b \in B \; a \, R \, b\}$
- The range of R is $\operatorname{ran} R = \{b \in B : \exists a \in A \; a \, R \, b\}$.
- The inverse of R is the relation $R^{-1} = \{(b, a) \in B \times A : (a, b) \in R\}$.
- Let S be a relation from B to C. The composition of S and R is the relation $S \circ R = \{(a, c) \in A \times C : \exists b \in B \; a \, R \, b \, S \, c\}$.

Properties of relations

Let R be a relation from A to B.

- The domain of R is $\operatorname{dom} R = \{a \in A : \exists b \in B \; a \, R \, b\}$
- The range of R is $\operatorname{ran} R = \{b \in B : \exists a \in A \; a \, R \, b\}$.
- The inverse of R is the relation $R^{-1} = \{(b, a) \in B \times A : (a, b) \in R\}$.
- Let S be a relation from B to C. The composition of S and R is the relation $S \circ R = \{(a, c) \in A \times C : \exists b \in B \; a \, R \, b \, S \, c\}$.

These should remind you of the definitions for functions. And indeed the same facts for functions also hold true for relations, e.g.
$R \circ (S \circ T) = (R \circ S) \circ T$.

Examples

Let's look at the relations $<$, $|$, and $\equiv \mod n$ on $\mathbb{N}$.

More properties of relations

Let R be a relation on A.

- R is reflexive if $a \, R \, a$ for all $a \in A$.
- R is symmetric if $a \, R \, b$ implies $b \, R \, a$ for all $a, b \in A$.
- R is transitive if $(a \, R \, b$ and $b \, R \, c)$ implies $a \, R \, c$ for all $a, b, c \in A$.

Examples

Let's look at the relations $<$, $|$, and $\equiv \mod n$ on $\mathbb{N}$.