Math 321: Intro to proofs

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Proofs

- Mathematics is a deductive discipline. To develop new truths we use logical tools. As such, when trying to see whether and why something is true, its logical structure informs how we go about that.
- And when trying to use a known truth, its logical structure informs how we can use it.
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- In the near future for this class we will investigate how the logical structure of mathematical statements yields certain proof strategies. We will break up this investigation by logical elements used.
- Some of this may be codifying things you already know and have internalized.

How to organize your thinking

- There's a lot of ways you can organize your thinking. Let me give you one.
- Divide your mathematical statements into two groups: knowns and goals. The knowns are statements you already know. These may be premises of a theorem, results already proved in class, definitions, and so on. On the other hand, the goals are the statements you want to prove. These are the places you want to reach.

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- The goal is to overtime transform things so that you can reach your goals from your knowns. The logical structure of statements helps with these transforms.

Proving If-thens

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Here's the idea: if P is false, then $P \to Q$ is true no matter what. So the only case we have to worry about is when P is true. In which case, we need to see that Q is also true to conclude that $P \to Q$ is also true.

Using If-thens

If you know $P \to Q$ and you also know P, then you can conclude Q. This rule is known as *modus ponens*.

knowns	goals
$P \to Q$	$\vdots$
P	

gets transformed into

knowns	goals
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Just from this one known we can't complete it. So let's remember some algebra, namely that if $0 < y$ and $a < b$ then $ay < by$. Also, since $<$ is transitive* we can conclude that indeed $0 < x$. This then gives us:

knowns	goals
$0 < x$	$x < x^2$
$1 < x$	
$(0 < x \land 1 < x) \to x < x^2$	

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So by *modus ponens* we know that $x < x^2$, establishing our goal.

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Let's write it up

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Theorem

Let x be a real number. If $1 < x$ then $x < x^2$.

Proof.

Assume that $1 < x$.* Then we can moreover conclude that $0 < x$ because $0 < 1$ and $<$ is transitive. By algebra facts, we know that $0 < x$ and $1 < x$ implies that $x < x^2$. So we conclude that $x < x^2$, completing the proof. □

* Phrases like "Assume that P" or "Suppose P" are used to signal to the reader that we're assuming the antesequent of a conditional so that we can prove the consequent.

Coda

Our write-up was really wordy. In pratice, mathematicians don't write things in so much detail. Indeed, for this statement a mathematician would probably just write something like "$1 < x$, so $x < x^2$" without even a hint of explanation, trusting that the reader understands this sort of basic algebra.

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But what we're doing now in this class is practicing proofs—understanding the strategies for coming up with proofs and practicing writing them up. So for those purposes, I do want you to give this overly detailed proofs. As you get more experience and move on to later classes you should feel free to not write in such minute detail.