Math 321: Introduction

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The scientific method

You probably learned this simplified picture for how scientists operate to produce knowledge.

1. Consider a natural phenomenon and make some preliminary observations.
2. Make a hypothesis about this phenomenon.
3. Design and carry out an experiment to test this hypothesis.
4. Analyze the results of the experiment.
5. Interpret this analysis to get a conclusion about the phenomenon.

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In mathematics, we produce knowledge differently.

The mathematical method

Mathematics is a deductive discipline. Rather than use experiments, our currency of knowledge is proofs—deductive arguments that take us step-by-step so that if we start out with true premises we always get true conclusions.

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A simplified picture of how mathematicians work:

1. Take an intuitive concept.
2. Mathematize the concept with a formal definition.
3. Make a conjecture about your definition.
4. Prove your conjecture, turning it into a theorem.
5. Conclude something about your concept.

An example of how to math: Alan Turing

Alan Turing

- Studied the concept of computability—what does it mean to compute an answer to a problem?
- Formalized this concept—now called the Turing machine.
- Proved there are some problems a Turing machine cannot compute the answer to.
- Conclusion: there are some problems a computer cannot compute an answer to.

The goal of this class

The goal of this class is to teach you the mathematical method so that you can go on to use it in future math classes. In lecture, your professors will state and prove theorems, and you need to be able to understand the underlying method to follow along. And for homework you will be asked to prove things.

The main focus is proofs. You should learn three things:

- How to read and understand proofs.
- How to construct your own proofs.
- How to write up a proof so that other mathematicians can understand your ideas.

We will also learn some basic language of mathematics—some concepts that appear over and over in different areas of math.

The purpose of proofs

Proofs do multiple things for us.

- They help us determine what is true.
- They help us determine why things are true.
- We can extract extra information by looking more closely at a proof.

An example: how many prime numbers are there?

Recall: A natural number $p > 1$ is prime if there are no natural numbers $x, y < p$ so that $xy = p$.

Question: Are there infinitely many primes?

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Cannot show the answer is no with examples. An example for no would be a number n so that no $p > n$ is prime. But that again would require looking at infinitely many examples.

So we need something else—a proof.

The idea for a proof

Infinite = not finite, so to show there are infinitely many primes we just have to show that any finite list of primes doesn't contain all of them.

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Example 1: start with 2, 3, and 5. We need to use them to produce a new prime. Try multiplying them and adding 1. We get 31, which is not divisible by 2, 3, nor 5 because it is 1 more than a multiple of them. And indeed 31 is a new prime.

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Example 2: start with 2 and 31. Again multiply them and add 1 to get 63. And $63 = 3^2 \cdot 7$. So we get two new primes, 3 and 7.

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We can turn this idea into a proof by writing up a general version. Instead of starting with a concrete finite list, we assume we have an arbitrary finite list and show this process produces new prime(s).

Writing up the proof

Theorem (Euclid)

There are infinitely many primes.

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Proof.

We will show that any finite list of primes cannot contain all primes. Suppose we have $p_1, p_2, \ldots, p_n$ an arbitrary finite list of distinct primes. Set

$$P = p_1 \cdot p_2 \cdot \cdots \cdot p_n.$$

Now consider $P + 1$. Like any natural number, $P + 1$ has a prime factorization. Let q be a prime in the prime factorization of $P + 1$. Note that q cannot be p_1 because then both P and $P + 1$ would be multiples of q. But that is impossible because $q > 1$. For the same reason, q cannot be p_i for any $i \leq n$. So we have seen that $p_1, p_2, \ldots, p_n$ is not a list of all primes, as desired. □

Extracting an algorithm from the proof

Our idea for the proof gives us an algorithm to generate new primes.

1. Start with a finite list of primes.
2. Multiply them and add 1.
3. Take the prime factorization of this number to get new primes.
4. Repeat as long as you want.

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A further question

The algorithm from the previous slide is really slow, since it produces large numbers we then have to factor. Can you come up with a faster algorithm to find new primes?