MATH321: INDUCTION PROOFS

KAMERYN J. WILLIAMS

Mathematical induction is a commonly used proof strategy. The purpose of this document is to give you some examples of some proofs by induction so you can see how they are written.

Before we see some examples I'd like to briefly discuss what the general template looks like. The basic format looks something like the following.

Theorem 1. *A statement $P(n)$ about natural numbers, usually though not always formulated as being about arbitrary natural numbers n. (Or k or a or whatever variable you want to use.) Other variables may also appear, but at least one of them is being universally quantified over.*

Proof. We prove this by induction on n. (Or if you used a different variable, that variable.)

The base case is $n = 0$. (It might be $n = 1$ or less often higher, depending on whether or not the theorem includes 0 as a possible case.) [Argument for base case goes here.]

For the inductive step, assume $P(n)$ and we want to show $P(n + 1)$. [Argument for this if-then goes here.] □

A few comments. First, it's good form to explicitly say at the beginning of your proof that you'll be using induction. This helps the reader to follow your argument. Second, it's best to say which variable you're doing induction on. This can get dropped if it's clear from context, but often a statement will include multiple variables and it's easier on the reader if they know which one is being used for the inductive argument. It's also good to mark the two parts of the inductive proof. When doing the inductive step, it's nice to the reader to make explicit where you use the inductive hypothesis of $P(n)$.

This is the general template, but you'll see variations. In particular, mathematicians will often omit easy details—often the base case is trivial—or the guidepost phrases which tell the reader what is happening where. For example, you might see something like:

- We prove this by induction on n. Assume $P(n)$. [Proceeds to show $P(n + 1)$.]

The base case was not written—presumably because it's easy—and nothing was said about how what was being written is the inductive step. I would advise you not to write induction proofs like this, at least in this class. Instead, it's better to be clearer and not skip steps.

1. Proofs by induction

Introductory remarks out of the way, let's see some examples. You can see further examples in Section 6.2 from the textbook.

Proposition 2. *The sum of the first k many odd numbers, for $k \geq 1$, is k^2.*

Date: November 4, 2020.

Proof. We prove this by induction on k. The base case $k = 1$ is the equation $1 = 1^2$, which is clear. For the inductive step, assume that

$$1 + \cdots + (2k - 1) = \sum_{i=1}^{k} 2i - 1 = k^2,$$

and we want to show that

$$1 + \cdots + (2k - 1) + (2k + 1) = \sum_{i=1}^{k+1} 2i - 1 = (k + 1)^2.$$

But this is straightforward: $(k + 1)^2 = k^2 + 2k + 1$ and this by inductive hypothesis is

$$\underbrace{1 + \cdots + (2k - 1)}_{=k^2} + (2k + 1).$$

□

Proposition 3. *For all positive integers k, we have $2^k > k$.*

Proof. We prove this by induction on k. The base case $k = 1$ is the equation $2^1 = 2 > 1$, which is obviously true. For the inductive step, assume that $2^k > k$ and we want to show that $2^{k+1} > k + 1$. We have that $2^k > k$, so $2^{k+1} = 2 \cdot 2^k > 2k = k + k > k + 1$. The last inequality is because $k \geq 1$. □

Proposition 4. *For all nonnegative integers k we have*

$$(*_k) \qquad 10^{k+1} - 1 = 9 \cdot 10^0 + 9 \cdot 10^1 + \cdots + 9 \cdot 10^k = \sum_{i=0}^{k} 9 \cdot 10^i.$$

Proof. We prove this by induction on k. The base case $k = 0$ is $10^1 - 1 = 10 - 1 = 9 = 9 \cdot 10^0$. For the inductive step, assume the equation $(*_k)$, and we want to show the equation $(*_{k+1})$. To that end, note that

$$10^{k+2} - 1 = (9 + 1) \cdot 10^{k+1} - 1 = 9 \cdot 10^{k+1} + 10^{k+1} - 1.$$

By inductive hypothesis, this is

$$9 \cdot 10^{k+1} + \left(\sum_{i=0}^{k} 9 \cdot 10^i \right) = \sum_{i=0}^{k+1} 9 \cdot 10^i.$$

This establishes $(*_{k+1})$, completing the proof. □

Exercise 5. Show for all $n \geq 2$ and all $k \geq 0$ that

$$n^{k+1} - 1 = \sum_{i=0}^{k} (n - 1) \cdot n^i.$$

[Hint: fix n and then do induction on k.]

Proposition 6. *The interior angles of a concave polygon with n sides sum to $(n - 2)\pi$ radians.*

Indeed, this is also true for non-concave polygons, but it's trickier to justify one step in the proof, so I restrict to the easier case.

Proof. The base case is $n = 3$. It is a well-known geometric fact that a triangle's interior angles sum to π radians.[1] Now proceed to the inductive step. Assume that every concave n-gon has its angles sum to $(n-2)\pi$ radians. Consider a concave $(n+1)$-gon. Because it is concave if you take the triangle formed from three of its adjacent vertices then this triangle is contained inside the $(n+1)$-gon. That is, we can split the $(n+1)$-gon into a triangle and an n-gon, where the sum of the interior angles for the $(n+1)$-gon is the sum of the interior angles of the triangle and the n-gon. By inductive hypothesis, the n-gon's interior angles sum to $(n-2)\pi$, and we know the triangle's interior angles sum to π. So the $(n+1)$-gon's interior angles sum to $(n-1)\pi$, which is exactly what we wanted to show. □

Remark 7. It's probably easiest to understand what's going on if you draw a picture: draw a concave polygon with > 3 sides and then form a triangle from three adjacent vertices.

For this next propositions, we need a new definition. Recall that $n!$ is the product of the integers from 1 to n, with $0! = 1$. Now define, for all n and $0 \leq k \leq n$:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

If $k < 0$ or $k > n$ then we let $\binom{n}{k} = 0$.

One could alternatively make the following definitions: $n!$ is the number of ways to arrange n objects in a linear order, while $\binom{n}{k}$ is the number of ways to pick k objects from a collection of size n, where you don't care about the order they were picked. A good exercise is to check that this definition gives the same formulae for $n!$ and $\binom{n}{k}$.

Proposition 8. *For all $n \geq 1$ and k we have $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$.*

This one we don't prove by induction :)

Proof. Fix n and k and consider $\binom{n+1}{k}$. Imagine picking k objects from a collection of size $n+1$. Split that collection into a subcollection of size n and an extra object singled out. There are two ways we might pick k objects from that $(n+1)$-sized collection. First, we might pick all k objects from the subcollection of size n. Second, we might pick the singled out object and then pick $k-1$ objects from the subcollection of size n. For the first, there are $\binom{n}{k}$ ways to make this choice. For the second, there are $\binom{n}{k-1}$ ways to make this choice. These two ways of picking are distinct, so there are in total $\binom{n}{k} + \binom{n}{k-1}$ ways of picking. That is, we have seen $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$. □

Remark 9. I didn't explicitly address the case $k < 0$ or $k > n$. Can you explain why thoes cases are fine?

Proposition 10. *For all $n \geq 0$ we have*

$$2^n = \sum_{k=0}^{n} \binom{n}{k}. \qquad (BT_n)$$

Proof. This we do by induction. I leave the base case $n = 0$ to you to check. For the inductive step, assume the equation (BT_n). Now consider

$$\sum_{k=0}^{n+1} \binom{n+1}{k} = \binom{n+1}{0} + \sum_{k=1}^{n+1} \binom{n+1}{k}.$$

[1] If you're unhappy with this, you're welcome to give your own proof of the fact :)

By the previous proposition we can rewrite this as

$$1 + \sum_{k=1}^{n+1} \left(\binom{n}{k-1} + \binom{n}{k} \right) = 1 + \left(\sum_{k=1}^{n+1} \binom{n}{k-1} \right) + \left(\sum_{k=1}^{n+1} \binom{n}{k} \right).$$

Let's re-index the left sum to be from $k = 0$ to $k = n$, and for the second sum include the $1 = \binom{n}{0}$ in the front and use that $\binom{n}{n+1} = 0$, so we can rewrite this as:

$$\left(\sum_{k=0}^{n} \binom{n}{k} \right) + \left(\sum_{k=0}^{n} \binom{n}{k} \right) = 2 \sum_{k=0}^{n} \binom{n}{k}.$$

By inductive hypothesis this is $2 \cdot 2^n = 2^{n+1}$, completing the proof. □

For the previous two propositions, it may help to think in terms of Pascal's triangle. Start with row 0 consisting of just the number 1. Then row 1 consists of two 1s, to each side of the 1 from row 0. For row $n + 1$, each entry is the sum of the two entries above, where a blank spot—for determining what goes on the two positions on the end of the row—is treated as a 0:

$$\begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ 1 \quad 4 \quad 6 \quad 4 \quad 1 \\ 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\ 1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\ \vdots \end{array}$$

If you index rows starting at 0, then $\binom{n}{k}$ is the k-th element on the n-th row. The first proposition says that each entry is the sum of the two above, and the second proposition says that the n-th row sums to 2^n.

Exercise 11. Prove the binomial theorem: for each natural number n, the binomial exponent $(A + B)^n$ can be written as

$$\binom{n}{0} A^n + \binom{n}{1} A^{n-1} B^1 + \cdots + \binom{n}{n-1} A^1 B^{n-1} + \binom{n}{n} B^n = \sum_{k=0}^{n} \binom{n}{k} A^{n-k} B^k.$$

[Hint: do induction on n.]

2. Definitions by recursion

Many definitions in mathematics are *recursive definitions*—you define how to handle the base case(s), and then inductively how to carry the definition upward. Often, these end up being by recursion on the natural numbers. You give the base case, usually $n = 0$ or $n = 1$, and then the recursive step defines the $n + 1$ case in terms of the n case.

As an example, let's go beyond exponentiation.

Definition 12. The *tetration*[2] operation, written $n \uparrow k$,[3] is defined as follows, for $n \in \mathbb{N}$:

- $n \uparrow 0 = 1$; and
- $n \uparrow k + 1 = n^{n \uparrow k}$.

Tetration grows really fast! To illustrate this, let's calculate the first few values of $2 \uparrow k$ and $3 \uparrow k$:

k	$2 \uparrow k$	$3 \uparrow k$
0	1	1
1	2	3
2	4	27
3	16	7,625,597,484,987
4	65,536	*[4]

Remark 13. The rules for exponents don't generalize to tetration. For example, $(xy) \uparrow n$ in general is not equal to $(x \uparrow n)(y \uparrow n)$.

Let's now make another definition by recursion, to see an example of where tetration can be applied.

Definition 14. The *iterated powerset* $\mathcal{P}^n(X)$ operation on a set X is defined as follows.

- $\mathcal{P}^0(X) = X$; and
- $\mathcal{P}^{n+1}(X) = \mathcal{P}(\mathcal{P}^n(X))$.

This is an instance of a more general phenomenon. If f is a function or operation (these are synonyms), then f^n is the function obtained by repeatedly doing f a total of n times.

Proposition 15. *For all natural numbers n we have $\mathcal{P}^{n+1}(\emptyset)$ has $2 \uparrow n$ elements.*

Proof. By induction. The base case $n = 0$ is the observation that $\mathcal{P}(\emptyset) = \{\emptyset\}$ has $1 = 2 \uparrow 0$ elements. For the inductive step, assume $\mathcal{P}^n(\emptyset)$ has $2 \uparrow n$ elements. To conclude that $\mathcal{P}^{n+2}(\emptyset) = \mathcal{P}(\mathcal{P}^{n+1}(\emptyset))$ has $2 \uparrow (n+1) = 2^{2 \uparrow n}$ elements it suffices to know that if a set X has k elements then $\mathcal{P}(X)$ has 2^k elements.

We have already discussed this in class, but let's do it again. Suppose X has k elements $x_1, \ldots, x_k$. To make a subset A of X you have k many binary choices: does A include the element x_i, for each $1 \leq i \leq k$. These choices are independent, so you have

$$\underbrace{2 \cdot 2 \cdot \cdots \cdot 2}_{n \text{ many}} = 2^n$$

possible subsets of X. □

The lesson of this proposition is that even if you start from a very small set—the empty set, the smallest set you can have—by iterating the powerset it grows very fast.

(Kameryn J. Williams) University of Hawaiʻi at Mānoa, Department of Mathematics, 2565 McCarthy Mall, Keller 401A, Honolulu, HI 96822, USA

E-mail address: kamerynw@hawaii.edu

URL: http://kamerynjw.net

[2] From "tetra", meaning 4. The idea is, if you start with addition as stage 1, then multiplication—iterated addition—is stage 2, exponentiation—iterated multiplication—is stage 3—and so tetration—iterated exponentiation—is stage 4.

[3] This is not standard notation, but there is no standard notation here.

[4] This overflowed my computer when I tried to calculate it. And this was with a programming language that would give me the exact value of $2 \uparrow 5 \approx 2 \cdot 10^{19728}$.