Math 321: Functions

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Relations

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We want to use similar ideas to formalize the notion of function.

Functions, intuitively

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Some possible answers:

- An expression in an independent variable x telling you how to produce a dependent variable y.
- A rule for transforming an input to an output.
- An algorithm for computing an output from some input.
- An association from inputs to outputs.

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We want a formalization that encapsulates all of these answers. The key commonality among them is that that inputs get assigned to outputs.

Functions, formally

Definition

A function f from a set A to a set B is a set of ordered pairs $(a, f(a))$ so that for each $a \in A$ there is a unique $f(a) \in B$ with $(a, f(a))$ in the function.
We write $f : A \to B$ to say that f is a function from A to B.

One way to think of this definition is it's like a giant lookup table: given an input $a \in A$ you look in the set of input-output pairs to find $f(a)$.

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You might want to tweak this definition.

- If we don't require $f(a)$ to be defined for all $a \in A$ we get a partial function.
- If we allow multiple values for $f(a)$ for a single $a \in A$ we get a multivalued function.

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Why not use one of these as the basic definition? They both get used in math, but the definition above has proved to be the most useful, so it gets taken as the basic one.

Functions, pictorally

Notation for functions

- Mathematicians usually prefer to write f to refer to the whole function, rather than something like $f(x)$.
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- The main reason is that it's easy to confuse $f(x)$ meaning the whole function with $f(x)$ meaning the value of the function at the point x.
- We can have multiple inputs to a function by having the domain be a set of pairs, or more generally n-tuples, e.g. $f : \mathbb{R}^2 \to \mathbb{R}$.
- For these, we prefer to write $f(a, b)$ rather than the ugly $f((a, b))$.

Definitions with functions

Let $f : A \to B$.

- The domain of f is $\operatorname{dom} f = \{a : f(a) \text{ is defined}\}$. (By definition, this is just A.)
- The range of f is $\operatorname{ran} f = \{b \in B : b = f(a) \text{ for some } a \in A\}$.
- If $g : B \to C$ then the composition of f and g is $g \circ f : A \to C$ defined as $(g \circ f)(a) = g(f(a))$.
- The identity function $\mathrm{id}_A : A \to A$ is defined as $\mathrm{id}_A(a) = a$ for all $a \in A$.
    - We sometimes write id if the domain is clear
- f is a constant function if $f(a)$ is the same for all $a \in A$.

One-to-one and onto

Let $f : A \to B$.

- f is one-to-one if $a_0 \neq a_1$ implies $f(a_0) \neq f(a_1)$ for all $a_0, a_1 \in A$. Equivalently: if $f(a_0) = f(a_1)$ implies $a_0 = a_1$ for all $a_0, a_1 \in A$. We also call f an injection.

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- f is onto B if for every $b \in B$ there is $a \in A$ so that $b = f(a)$. We also call f a surjection onto B.
- If f is both one-to-one and onto B we call f a bijection onto B.

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Suppose $f : A \to B$.

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- Another problem: what if $b = f(a_0) = f(a_1)$? Do we define $g(b)$ to be a_0 or a_1?
  This is not so easily patched. We can just say g is a multivalued function, but there's no reasonable way to turn it into a function in general.

Instead, we have to put a restriction on f: it should be one-to-one.

Inverses

Suppose $f : A \to B$ is one-to-one.

Definition

The inverse of f is the function $f^{-1} : \operatorname{ran} A \to A$ defined as $f^{-1}(b) = a$ iff $f(a) = b$.

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Some facts:

- If f is onto B then $f^{-1} : B \to A$.
- $\operatorname{ran} f = \operatorname{dom} f^{-1}$ and $\operatorname{dom} f = \operatorname{ran} f^{-1}$.
- $f^{-1} \circ f$ is the identity function on A and $f \circ f^{-1}$ is the identity function on $\operatorname{ran} A$.