Math 321: Countable and uncountable sets

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Last time

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- For all $b \in B$ there is a unique $a \in A$ so that $f(a) = b$.

We used this notion to give a definition of when two sets have the same size: sets A and B are equinumerous if there is a bijection $f : A \to B$.

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Definitions for cardinality

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- $|A| = |B|$ means that there is a bijection $f : A \to B$.
- $|A| \leq |B|$ means that there is a one-to-one function $f : A \to B$.
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  The proof of this theorem is reasonably involved. Due to time constraints we have to skip it :(
  You can find a proof in section 7.3 of the textbook.
- The lesson: if you want to show there is a bijection between A and B it is enough to find one-to-one functions $A \to B$ and $B \to A$.

An example

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More generally, any two nondegenerate intervals have the same cardinality.

Countable and uncountable sets

- Say that a set A is countable if $|A| \leq |\mathbb{N}|$. That is, A is countable if there is a one-to-one function $f : A \to \mathbb{N}$.
- Every finite set is countable.
- We say countably infinite to distinguish countable, infinite sets from finite sets.
- If A is not countable we call it uncountable.

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(base case) Since A is nonempty, we simply pick any element of A to assign to be $f(0)$.

(induction step) We have already defined $f(0), \ldots, f(n)$. It cannot be that $A = \{f(0), \ldots, f(n)\}$, as if that were the case we would have that A is finite and hence countable, whereas we know A is uncountable. In other words, $A \setminus \{f(0), \ldots, f(n)\}$ is nonempty. So pick some element of this set to assign to be $f(n+1)$.

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We can always continue, so we have a one-to-one function $f : \mathbb{N} \to A$. □

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Theorem (Cantor)

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We have to show there is no one-to-one function $f : \mathbb{R} \to \mathbb{N}$, which we do by contradiction.

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- The answer is that you can use this fact to prove theorems.
- If you show that $A \subseteq \mathbb{R}$ is a countable subset of the reals, then you can conclude that there are reals not in A.
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- The answer is that you can use this fact to prove theorems.
- If you show that $A \subseteq \mathbb{R}$ is a countable subset of the reals, then you can conclude that there are reals not in A.
- (More generally, if $A \subseteq B$ and $|A| < |B|$ then $B \setminus A$ is nonempty.)
- For example, say that a real number is algebraic if it is a root of a polynomial with rational coefficients. If a real number is not algebraic we call it transcendental.
  - For example, $\sqrt{2}$ is algebraic, as seen by the polynomial $x^2 - 2$.
  - This is a nontrivial result to prove, but π is transcendental.
- Showing a particular number is transcendental is quite hard, but you can show that transcendental numbers exist by showing that the set of algebraic numbers is countable. (This is one of the final exam problems.) In fact, Cantor originally proved that $\mathbb{R}$ is uncountable as a lemma in a new proof for the existence of transcendental numbers.

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Theorem (Cantor)

$\mathcal{P}(\mathbb{N})$ is uncountable. More generally, if A is any set then $|A| < |\mathcal{P}(A)|$.

Proof.

We can see that $|A| \leq |\mathcal{P}(A)|$ by looking at the one-to-one function $s : A \to \mathcal{P}(A)$ defined as $s(a) = \{a\}$. So we just have to see that there is no bijection $f : A \to \mathcal{P}(A)$, which we do by contradiction.

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Suppose $f : A \to \mathcal{P}(A)$ is a bijection. Consider $D = \{a \in A : a \notin f(a)\}$, a subset of A. Since f is a bijection, there is $d \in A$ so that $f(d) = D$. Let's now consider two cases.

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(Case 1: $d \in D$) By definition of D, we get that $d \notin f(d) = D$, a contradiction.

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Suppose $f : A \to \mathcal{P}(A)$ is a bijection. Consider $D = \{a \in A : a \notin f(a)\}$, a subset of A. Since f is a bijection, there is $d \in A$ so that $f(d) = D$. Let's now consider two cases.
(Case 1: $d \in D$) By definition of D, we get that $d \notin f(d) = D$, a contradiction.
(Case 2: $d \notin D$) By definition of D, we get that $d \in f(d) = D$, a contradiction.
Either way we get a contradiction, so there can be no such bijection f. □

Some equinumerosities for uncountable sets

The following sets all have the same cardinality:

- $\mathbb{R}$;
- Any nondegenerate interval (a, b), $[a, b]$, $(a, b]$, or $[b, a)$;
- $\mathcal{P}(\mathbb{N})$.

Cardinalities of infinite sets

- Sets are linearly ordered by cardinality: for two sets A and B, either $|A| < |B|$, $|A| = |B|$, or $|A| > |B|$.
- Moreover, sets are well-ordered by cardinality. In particular, if you have an infinite set A there is a smallest cardinality $> |A|$.

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- We use $\aleph_0$ (the Hebrew letter aleph) for the smallest cardinality of an infinite set. That is, $\aleph_0 = |\mathbb{N}|$.
- And $\aleph_{n+1}$ is the smallest cardinality $> \aleph_n$. And we can continue this upward transfinitely, beyond just the finite indices.
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- Given a set A, we write $2^{|A|}$ for $|\mathcal{P}(A)|$.

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- You should've gotten the answer $|B|^{|A|}$.
- $2^{|A|}$ is the cardinality of the set of functions from A to a two-element set.
- Think: a subset $X \subseteq A$ is really a function mapping each element of A to either `yes` or `no`.

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- There are extra axioms you can add which do settle the continuum hypothesis, some saying yes and others saying no. But as yet no axiom has been proposed which settles CH and which the experts accept.
- But I won't be able to talk about such in this class, since that is a graduate-level topic.