MATH655 EXERCISES: TRANSFINITE RECURSION

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Here are some exercises for practice with transfinite recursion.

Definition 1. Say that a sequence $\langle \alpha_\xi : \xi \in \gamma \rangle$, where γ is either an ordinal or the class Ord of all ordinals, is *normal* if it is increasing and continuous. That is, $\alpha_\xi \leq \alpha_{\xi+1}$ and for limit $\lambda < \gamma$ we have that $\alpha_\lambda = \sup_{\xi<\lambda} \alpha_\xi$.

Exercise 2. Show that any Ord-length normal sequence $\langle \alpha_\xi : \xi \in \mathrm{Ord} \rangle$ has arbitrarily large fixed points, i.e. there are arbitrarily large ξ so that $\alpha_\xi = \xi$.

Exercise 3. One way to prove Zorn's lemma is through transfinite recursion. Assume that $(P, <)$ is a partially ordered set with the property that every chain in P has an upper bound in P. Let $\langle x_\alpha : \alpha < \beta \rangle$ be a sequence enumerating P. Use transfinite recursion along β to produce a maximal element of P.

One important application of transfinite recursion is to prove the existence of the satisfaction relation for structures. Let us specialize to the case of structures which are transitive sets equipped with the membership relation.

Definition 4. If t is a transitive set, φ is a formula in the language of set theory, and $\bar{a}$ are elements of t, we write $t \models \varphi(\bar{a})$ if $\varphi(\bar{a})$ is true in t, with the symbol for the membership relation being interpreted as $\in$ restricted to t.[1]

More formally, this *satisfaction relation* is defined according to the following Tarskian recursion:

(1) $t \models a = b$ iff $a = b$;
(2) $t \models a \in b$ iff $a \in b$;
(3) $t \models \varphi \wedge \psi$ iff $t \models \varphi$ and $t \models \psi$;
(4) $t \models \varphi \vee \psi$ iff $t \models \varphi$ or $t \models \psi$;
(5) $t \models \neg\varphi$ iff it's not the case that $t \models \varphi$;
(6) $t \models \forall x \varphi(x)$ iff for all $a \in t$ we have $t \models \varphi(a)$;
(7) $t \models \exists x \varphi(x)$ iff there is $a \in t$ so that $t \models \varphi(a)$;

Exercise 5. Show that every transitive set t admits a unique satisfaction relation. (Hint: this is a one line proof.)

Exercise 6. One might try to use transfinite recursion to carry out the same recursive definition along the structure $(V, \in)$ to produce a class satisfaction relation $V \models \varphi(\bar{a})$. But this won't work. Prove Tarski's theorem on the undefinability of truth, that there is no class with this property. Explain why Tarski's theorem does not contradict the principle of transfinite recursion. Why is it that transfinite recursion doesn't let us produce the satisfaction relation for V?

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[1] For those with a background in logic, this is the usual model theoretic satisfaction relation for the structure $(t, \in)$.

Exercise 7. Show that if $(D, <)$ is a countable dense linear order without endpoints, then $(D, <) \cong (\mathbb{Q}, <)$. (Hint: well-order D and $\mathbb{Q}$ in ordertype ω and use that to recursively build an isomorphism between them.)

Exercise 8. Let E and F be extensional, well-founded graphs with set-sized domains. A *partial isomorphism* from E to F is a partial function π from E to F so that $a \mathrel{E} b$ iff $\pi(a) \mathrel{F} \pi(b)$ for $a, b \in \operatorname{dom} \pi$. An *initial partial isomorphism* from E to F is a partial isomorphism whose domain and range are both downward closed—if $a \in \operatorname{dom} \pi$ and $b \mathrel{E} a$ then $b \in \operatorname{dom} \pi$, and similarly for range.

Show that there is a unique maximum initial partial isomorphism π from E to F. That is, show there is a unique initial partial isomorphism π from E to F so that if σ is any initial partial isomorphism from E to F then $\sigma \subseteq \pi$.

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