MATH655 LECTURE NOTES: PART 3 LARGE CARDINALS AND FORCING

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In this, the final part of this course, we will survey some connections between large cardinals and forcing.

1. Large cardinals cannot settle the continuum hypothesis

We have seen that ZFC does not settle the value of $2^{\aleph_0}$. Both CH and $\neg\mathsf{CH}$ are consistent with ZFC. Of course, adding new axioms may decide the continuum hypothesis. Trivially, adding CH as an axiom settles the question. But there are less trivial axioms one could add. For example, Gödel's axiom of constructibility $V = L$ implies GCH. On the other side, the proper forcing axiom PFA implies that $2^{\aleph_0} = \aleph_2$.

A natural question is whether large cardinal axioms can settle CH. This is natural for a few reasons. First off, large cardinal axioms are the extensions of ZFC we have covered in this class. So they are of interest for that reason :P More substantially, large cardinal axioms are generally considered to be the strongest candidates for axioms extending ZFC. I shan't dive into the arguments for this, as that would require going too deep into philosophy of mathematics. And note that this opinion is by no means universal, and there are well-regarded set theorists who don't ascribe to it. But we will take it as a given that large cardinal axioms are of interest. To gesture at the historical importance of these questions, Gödel had hoped that large cardinals would be able to settle CH.

How unfortunate then that large cardinals cannot decide CH.

This statement must be qualified a bit. There isn't a precise formal definition of a large cardinal. And "definitions" which are too broad admit silly counterexamples.

Exercise 1. Come up with a property a cardinal can have so that the existence of such a cardinal implies both $\mathrm{Con}(\mathsf{ZFC})$ and CH.

But the statement is true for all established large cardinal notions. Let us begin by seeing why the existence of an inaccessible cardinal cannot settle CH.

Observation 2. *Suppose κ is inaccessible. Then, after forcing forcing with either $\mathbb{P} = \mathrm{Add}(\omega, \omega_2)$ or $\mathbb{Q} = \mathrm{Add}(\omega_1, 1)$, we have that κ remains inaccessible.*

Proof. We have already seen that forcing with a poset cannot affect the continuum function above its cardinality. Since $\mathbb{P}$ and $\mathbb{Q}$ are both much smaller than κ, we thereby get that κ remains strong limit in the extension. So we only have to see that κ is regular. Fix regular $\lambda < \kappa$ so that $\mathbb{P}$ and $\mathbb{Q}$ both have the λ-cc. ($\lambda = \beth_1^+$ suffices.) Then $\mathbb{P}$ and $\mathbb{Q}$ preserve regular cardinals $\geq \lambda$, so we are done. $\square$

Corollary 3. *The same holds for κ being Mahlo.*

Proof. The only remaining step is to see that forcing with $\mathbb{P}$ or $\mathbb{Q}$ doesn't change whether a subset of κ is stationary. Exercise: check this! $\square$

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Let me note that what we used about $\mathbb{P}$ and $\mathbb{Q}$ here is that they are both small relative to κ. In general, small forcings cannot destroy inaccessibility. So any of the independence results about small sets which can be forced on or off by small forcing will not be settled by inaccessible cardinals.

Moving to larger forcings, we can destroy the inaccessibility of κ when changing the cardinality of the continuum. For instance, if we force with $\mathrm{Add}(\omega, \kappa)$ then κ will no longer be strong limit, hence no longer inaccessible. On the other hand, something of the large cardinal nature of κ will remain.

Proposition 4. *Let κ be inaccessible and λ be any uncountable cardinal. Then κ remains weakly inaccessible after forcing with* $\mathrm{Add}(\omega, \lambda)$.

Proof. We saw in part 2.2 that $\mathrm{Add}(\omega, \lambda)$ has the ccc, so it will preserve cardinals and cofinalities. In particular, κ remains regular and remains an aleph fixed-point. $\square$

Let us move now from inaccessible cardinals to something more interesting. Here, we will say that a forcing $\mathbb{P}$ is *small* when $|\mathbb{P}| < \kappa$, for a specific κ we have in mind. As before, what we want to see is that if κ has a large cardinal property then small forcing won't destroy that large cardinal property. It will immediately follow that that large cardinal property cannot settle CH, or many other independence results about small cardinals.

We first look at measurable cardinals.

Theorem 5 (Lévy–Solovay). *Let κ be a cardinal and $\mathbb{P}$ be small. Then κ is measurable in the forcing extension iff it was measurable in the ground model.*

Proof. We really prove the following two facts, which amount to an elaboration of the statement of the theorem. Here, D is any set in V and $G \subseteq \mathbb{P}$ is V-generic.

(1) Each κ-complete ultrafilter on D in V generates a κ-complete measure on D in $V[G]$.
(2) Each κ-complete ultrafilter on D in $V[G]$ is generated in this way by a measure in the ground model.

(1) Let U be a κ-complete ultrafilter on D in V. Defined W in $V[G]$ as

$$W = \{X \subseteq D : \exists Y \in U \; Y \subseteq X\}.$$

It is straightforward to see that W is a filter. (This is an instance of the more general phenomenon of a filter base generating a filter.) Next we want to see it is an ultrafilter. To that end, fix $X \subseteq D$ in $V[G]$ and suppose $p \Vdash \dot{X} \notin \dot{W}$ for some $p \in G$. But then, for any $q \leq p$, the set $X_q = \{a \in D : q \Vdash \check{a} \in \dot{X}\}$ cannot be in U. Because $|\mathbb{P}| < \kappa$ and U is κ-complete, we get that $Y = \bigcup_{q \leq p} X_q$ is not in U. Because U is an ultrafilter, it must be that $D \setminus Y \in U$. But $X = \bigcup_{q \in G} X_q \subseteq Y$, so $D \setminus Y \subseteq D \setminus X$, so $D \setminus X \in W$.

Next let us see that W is κ-complete. To this end, suppose we have $X_\alpha \notin W$ subsets of D in $V[G]$ for each $\alpha < \gamma < \kappa$. Suppose $p \in G$ forces this. Similar to before, for each $q \leq p$ and each $\alpha < \gamma$ set $X_{\alpha,q} = \{a \in D : q \Vdash \check{a} \in \dot{X}_\alpha\}$. We then get, similar to before, that each $X_{\alpha,q} \notin U$. So by κ-completeness we get that $\bigcup_{\alpha<\gamma} \bigcup_{q \leq p} X_{\alpha,q} \notin U$. Since this union covers $X = \bigcup_{\alpha<\gamma} X_\alpha$, we therefore get $X \notin W$, as desired.

(2) Suppose $W \in V[G]$ is a κ-complete ultrafilter on $D \in V$, and set $U = W \cap V$. Let us see that U is a κ-complete ultrafilter in V and that W is generated by U as in (1). For the latter claim, we can decompose $X \in W$ as $X = \bigcup_{q \in G} X_q$ as before, where $X_q = \{a \in D : q \Vdash \check{a} \in \dot{X}\}$. So by κ-completeness of W we must have $X_q \in W$ for some q. But $X_q \in V$, so $X_q \in U$. So W consists of supersets of sets in U, just as we wanted to check. From this it immediately follows that U is an ultrafilter.

Finally, we must see that U is κ-complete. Start by considering $p \in G$ so that $p \Vdash \dot{W}$ is a $\check{\kappa}$-complete measure on $\check{D}$. Now in V, for each $q \leq p$ set $F_q = \{X \subseteq D : q \Vdash \check{X} \in \dot{W}\}$. Then F_q is is a κ-complete filter on D. (Exercise: check this!) But note that $U = \bigcup\{F_q : q \leq p \text{ and } q \in G\}$. Because W is generated by U, it follows that $W = \bigcup\{\langle F_q \rangle^{V[G]} : q \leq p \text{ and } q \in G\}$, where $\langle B \rangle$ is the filter generated by the filter base B.[1] Suppose $W \neq \langle F_q \rangle^{V[G]}$ for each q. But then we can pick $X_q \in W \setminus \langle F_q \rangle^{V[G]}$ for each q, and by κ-completeness get than $\bigcap_q X_q \in W$ but is not in any $\langle F_q \rangle^{V[G]}$. This is a contradiction, so it must be that $W = \langle F_q \rangle^{V[G]}$ for some $q \in G$. But then $U = \langle F_q \rangle^{V[G]} \cap V = F_q \in V$. So U is κ-complete, because F_q is κ-complete. □

This settles the question of whether measurable cardinals can settle CH. But let's pursue this line of investigation further. With measurable cardinals, what we were really interested in was the corresponding elementary embeddings of V they gave. We've seen that small forcings have a transparent effect on measures. Can we get something similar for the ultrapower embeddings generated by the measures?

The answer of course is yes. We wouldn't spend time on it if it weren't going to lead anywhere.

Definition 6. Let $V \subseteq V^*$ be models of set theory, and $j : V \to M$ and $j^* : V^* \to M^*$ are embeddings. Say that j *lifts* to j^* if $j = j^* \restriction V$.

Theorem 7. *Every embedding $j : V \to M$ lifts uniquely to an embedding $j^* : V[g] \to M[j(g)]$ in any forcing extension $V[g]$ via a V-generic $g \subseteq \mathbb{P}$ with $\mathbb{P}$ small relative to* $\operatorname{crit} j$.

Proof. Because $|\mathbb{P}| < \kappa = \operatorname{crit} j$ we have that $j(\mathbb{P}) = j''\mathbb{P}$ and thus $j \restriction \mathbb{P} : \mathbb{P} \to j(\mathbb{P})$ is an isomorphism. Set $h = j''g$. Since h is the image of g under an isomorphism of posets, we get that h is V-generic and, since $M \subseteq V$, is moreover M-generic. Now define j^* as $j(\tau_g) = j(\tau)_h$. I claim this map is well-defined. To see this, suppose $\tau_g = \sigma_g$. Then there is some condition $p \in g$ so that $p \Vdash_{\mathbb{P}} \tau = \sigma$. So $j(p) \Vdash_{j(\mathbb{P})} j(\tau) = j(\sigma)$. Since $j(p) \in h$ we therefore get $j(\tau)_h = j(\sigma)_h$. By a similar argument we get that j^* is one-to-one.

Now observe that $j^* : V[g] \to M[h]$, because $j(\tau) \in M$. To see that this map is an elementary embedding, suppose $V[g] \models \varphi(\tau_g)$. Then there is $p \in g$ with $p \Vdash_{\mathbb{P}} \varphi(\tau)$ in V, and so $(p) \Vdash_{j(\mathbb{P})} \varphi(j(\tau))$ in M. But then $M[h] \models \varphi(j(\tau)_h)$. Next, note that if $x \in V$ then $x = \check{x}_g$ and so $j^*(x) = j^*(\check{x}_g) = \check{j(x)}_h = j(x)$. So $j^* \restriction V = j$.

Finally, we see that j^* is unique. Note that any lift of j must send g to $j''g = h$. But then it must send τ_g to $j(\tau)_h$, so it must be j^*. □

Because $\mathbb{P}$ is small, taking an isomorphic copy as necessary we may assume that $\mathbb{P} \in V_\kappa$. But then $j(\mathbb{P}) = \mathbb{P}$ and j fixes $\mathbb{P}$ pointwise. In particular, $j''g = g$ and so the lifted embedding is of the form $j^* : V[g] \to M[g]$.

Theorem 8. *Suppose $U \in V$ is a κ-complete measure on D and suppose $\mathbb{P} \in V_\kappa$ is small. Then the unique lift of the corresponding ultrapower embedding $j_U : V \to M$ is the ultrapower j_W by the measure W generated by U.*

Proof. We have seen that the lift of j_U is unique, so we only need to see that j_u lifts to j_W. Observe that j_W has the form $j_W : V[g] \to N[j_W(g)]$ where $N = \bigcup_\alpha j_W(V_\alpha) = \bigcup j_W{}''V$. We will see that $M = N$ and that $j_U = j_W \restriction V$. First, define $\pi : M \to N$ as follows. Given $[f]_U \in M$ let $\pi([f]_U) = [f]_W$. This map is well-defined, because if $\{a \in D : f(a) = f'(a)\} \in U$ then it is in

[1] That is, $X \in \langle B \rangle$ iff $X \supseteq b$ for some $b \in B$.

W and so $[f]_W = [f']_W$. And a similar argument shows that π preserves membership, so it's a homomorphism. Let us see π is onto. Take a function $f : D \to V$ in $V[g]$ and let $\dot{f}$ be a name for f. Take $p \in g$ so that $p \Vdash \dot{t} : \check{D} \to \check{V}$. Now define f_q for each $q \leq p$ as $f_q(a) = b$ iff $q \Vdash \dot{f}(\check{a}) = \check{b}$. Then $f = \bigcup_{q \in g} f_q$. Because $D = \bigcup_{q \in g} \operatorname{dom} f_q$ and $|g| < \kappa$ we get that $\operatorname{dom} f_q \in W$ for some $q \in g$. But then, on a W-big set we have that $f = f_q$, and so $[f]_W = [f_q]_W$. Now, since $f_q \in V$, we get that $\pi([f_q]_U) = [f_q]_W = [f]_W$. So π is onto. And we can see that π is one-to-one because if f and f' differ on a set in U then they must differ on a set in W. Thus π is an isomorphism. Therefore, since the membership relation is rigid, we get that π is the identity so $M = N$. Finally, if $x \in V$ then $j_U(x) = [c_x]_U = \pi([c_x]_U) = [c_x]_W = j_W(x)$. So $j_u = j_W \restriction V$. □

Corollary 9. *If W is a κ-complete measure in a small forcing extension $V[g]$ then the corresponding ultrapower embedding j_W is the lift of an ultrapower embedding in V.*

Proof. We may assume without loss that W is a measure on a set D in V, by taking a bijective onto an ordinal if necessary. So then $U = W \cap V$ is a κ-complete measure in V on D. And then j_U lifts to j_W. □

On the other hand, this does not generalize if we consider embeddings more general than ultrapower embeddings.

Exercise 10. Let $V[x]$ be an extension of V by adding a Cohen real x. Suppose U_0 and U_1 are normal measures on κ in V, which generate W_0 and W_1 in $V[x]$. In $V[x]$ consider the direct limit $j : V[x] \to M[x]$ of the iterated ultrapower embeddings $j_k : V[x] \to M[x]$ defined by $j_k = j_{W_{i_k}} \circ \cdots \circ j_{W_{i_0}}$, where $i_n = x(n)$. That is, we choose whether to use W_0 or W_1 by looking at the bits of x and iterate this out ω many times. Show that x can be reconstructed from $j \restriction V$. Conclude that j cannot be the lift of an embedding from V.

Let us turn now to supercompact cardinals. Much of the work has already been done.

Theorem 11. *Supercompact cardinals are preserved by small forcing. More specifically, any supercompactness embedding $j : V \to M$ in V lifts uniquely to a supercompactness embedding $j : V[g] \to M[j(g)]$ in a small forcing extension $V[g]$.*

Proof. We already know that a λ-supercompactness embedding j lifts uniquely to an $j : V[g] \to M[j(g)]$, since supercompactness embeddings are ultrapower embeddings. We just have to see that j is a λ-supercompactness embedding. Recall that $j''\lambda$ is a seed which generates all of M. We want to see that $j''\lambda$ is a seed which generates all of $M[j(g)]$, as that is the one missing piece.

We have that every element of M is of the form $j(f)(j''\lambda)$ for some function $f \in V$. Because elements of $M[j(g)]$ are of the form $\tau_{j(g)}$ for some name $\tau \in M$, we get that elements of $M[j(g)]$ are of the form $j(f)(j''\lambda)_{j(g)}$. In $V[g]$, define a new function f_0 with the same domain as f as $f_0(x) = f(x)_g$ if $f(x)$ is a name, and $f_0(x) = 4$ otherwise. Observe now that $j(f_0)(j''\lambda) = j(f)(j''\lambda)_{j(g)}$ whenever this is a name. So every element of $M[j(g)]$ has the form $j(f_0)(j''\lambda)$. That is, $j''\lambda$ is a seed which generates all of $M[j(g)]$. □

The argument that $j''\lambda$ was a seed for the lifted embedding generalizes.

Exercise 12. Suppose a set of seeds S generates an embedding $j : V \to M$. Show that if $j : V[g] \to M[j(g)]$ is a lift of j then S generates the lifted embedding.

We also get that the lifted embedding is the ultrapower by the measure W generated by the measure U in the ground model giving rise to the original embedding $j : V \to M$. In particular, this tells us that W is fine and normal.

Observe that in our argument about $j''\lambda$ being a seed for $M[j(g)]$ we didn't need that the forcing was small. We only used smallness of the forcing to know that there was a lift. So we get the following.

Corollary 13. *Suppose a λ-supercompactness $j : V \to M$ lifts to $j : V[G] \to M[j(G)]$ in a forcing extension. Then the lifted embedding is a λ-supercompactness embedding in the extension.* □

Put differently, if we want to know that κ remains λ-supercompact after forcing, we only have to check that the embedding lifts. Once we know that, we get for free that it is a λ-supercompactness embedding. That's nice :)

To close off this section, let us briefly dip our toes into forcings that aren't small. While it's nice to know that small forcing is not destructive, often we find ourselves in a situation where we need to do a larger forcing. For example, we may want to know whether we can force the GCH to hold at κ. This cannot be done with small forcing. So we want criteria for when we preserve the large cardinal properties of κ.

Observe that we do need something. If we allow arbitrary forcings of any size, then we can destroy the large cardinal properties of κ. For instance, force with $\mathrm{Col}(\omega, \kappa)$.

Lemma 14 (The lifting criterion)**.** *Suppose $j : M \to N$ is an elementary embedding between models of set theory (set- or class-sized), with forcing extension $M \subseteq M[G]$ and $N \subseteq N[H]$ via posets $\mathbb{P}$ and $j(\mathbb{P})$. Then j lifts to an embedding $j^* : M[G] \to N[H]$ with $j^*(G) = H$ iff $j''G \subseteq H$. In such a case, j^* is unique.*

Proof. ($\Rightarrow$) This is immediate. If $j^*(G) = H$ then $j''G = j^{*\prime\prime}G \subseteq j^*(G) = H$.

($\Leftarrow$) This is the substantive direction. We must define j^*. First, observe that every set in $M[G]$ is of the form τ_G where $\tau \in M^{\mathbb{P}}$. This suggests how to define j^*: set $j^*(\tau_G) = j(\tau)_H$. This is well-defined, since if $\tau_G = \sigma_G$ this is forced by some $p \in G$ and so $j(p) \Vdash_{j(\mathbb{P})} j(\tau) = j(\sigma)$. But since $j(p) \in H$, we get that $j(\tau)_H = j(\sigma)_H$. And the same argument establishes that j^* preserves membership, and indeed is elementary, but using $p \Vdash_{\mathbb{P}} \varphi(\sigma, \ldots)$ instead. In particular, if $\Gamma = \{(\check{p}, p) : p \in \mathbb{P}\}$ is the canonical name for the generic, then $j^*(G) = j^*(\Gamma)_H = H$. We also get that $j^*(x) = j^*(\check{x}_G) = j(\check{x})_H = \check{j(x)}_H = j(x)$. So j^* lifts j. Finally, since any lift j^* of j must satisfy $j^*(\tau_G) = j(\tau)_{j^*(G)}$, it is clear that the embedding is uniquely determined by $j^*(G) = H$. □

I want to highlight that while the lift j^* is the unique lift satisfying $j^*(G) = H$, this does not mean that j^* must always be the unique lift. It is possible for j to lift to other embeddings $j^\dagger$, but with $j^\dagger(G) = H^\dagger \neq H$ yet nevertheless $N[H^\dagger] = N[H]$.

Theorem 15. *Suppose $j : V \to M$ is an ultrapower embedding in V via a measure U on κ. Let $\mathbb{P}$ be κ^+-closed and $G \subseteq \mathbb{P}$ be V-generic. Then j lifts uniquely to an embedding $j : V[G] \to M[j(G)]$ in the extension.*

Proof. Because G is a filter, elements of G are directed. So by elementarity of j we get that $j''G$ is a directed subset of $j(\mathbb{P})$. Let $H = \{q \in j(\mathbb{P}) : \exists p \in G \; j(p) \leq q\}$. Then H is a filter. (Exercise: check this!) We want to see that H is M-generic for $j(\mathbb{P})$. Take $D \subseteq j(\mathbb{P})$ an open dense set in M. Then $D = j(d)(\kappa)$, where $d : \kappa \to V$ is a function in V. So this means that $A = \{\alpha < \kappa : j(\alpha)$ is an open dense subset of $\mathbb{P}\}$ is in U. Let $\langle \alpha_i : i < \kappa \rangle$ enumerate A in increasing order. Now let's define a descending sequence of conditions $\langle p_i : i \leq \kappa \rangle$ in G. Start with $p_0 \in G$ arbitrary. Given p_i, let $p_{i+1} \leq p_i$ be in $G \cap d(\alpha_i)$. At limit stages $\ell \leq \kappa$, take $p_\ell \in G$ a lower bound to the partial sequence so far which. At the end, we have $p = p_\kappa \in G$ so that $p \in d(\alpha)$ for U-many $\alpha < \kappa$. So we get that $j(p) \in j(d)(\kappa) = D$. Thus, since $j(p) \in j''G \subseteq H$ we get that $H \cap D \neq \emptyset$.

Now, by the lifting criterion we get that j lifts to $j : V[G] \to M[j(G)]$ with $j(G) = H$. To see the lift is unique, suppose $j^* : V[G] \to M[j^*(G)]$ is another lift of j. It must then be that $j''G = j^{*''}G \subseteq j^*(G)$. So by the definition of H, we get that $H \subseteq j^*(G)$. But then $H = j^*(G)$. So by the uniqueness property of the lifting criterion we get $j = j^*$. $\square$

Note that we don't need this machinery to conclude that κ^+-closed forcing preserves the measurability of κ. Since κ^+-closed forcing doesn't add new subsets of κ, all the old measures remain measures in the extension. This theorem gives us extra information, telling us that the embeddings in the ground model lift to the embedding.

Corollary 16. *Suppose κ is measurable. Then you can force $2^\kappa = \kappa^+$ while preserving the ultrapower embeddings from measures on κ.*

Proof. Force with $\mathrm{Add}(\kappa^+, 1)$. We saw in part 2.2 that this forces $2^\kappa = \kappa^+$. And since it is κ^+-closed, we know that measures on κ in the ground model lift. $\square$

2. The number of normal measures

Recall that if κ is measurable then there is a normal measure on κ. One question that has been investigated by set theorists is how many normal measures there may be. Using an *inner model theoretic* argument, Kunen showed that it is consistent for κ to be measurable and have a unique normal measure. We won't see Kunen's argument, as that would require we devote several weeks to the basics of inner model theory. But it gives us a jumping off point. Can we have more than one normal measure on κ?

Kunen and Paris did better than > 1, and showed that it is consistent to have as many normal measures as possible.

Theorem 17 (Kunen–Paris)**.** *Suppose κ is measurable. Then there is a forcing extension in which κ has 2^{2^κ} many normal measures.*

Proof. Without loss of generality we may assume $2^\kappa = \kappa^+$. The poset $\mathbb{P} = \mathbb{P}_\kappa$ we force with will be an Easton support iteration on the inaccessible cardinals $\gamma < \kappa$, with the γth stage being $\mathbb{Q}_\gamma = \mathrm{Add}(\gamma^+, 1)^{V[G_\gamma]}$, where $G_\gamma \subseteq \mathbb{P}_\gamma$ is the restriction of the generic G to the stages below γ. Let $G \subseteq \mathbb{P}_\kappa$ be the V-generic for the forcing. Note that $j(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \dot{\mathbb{P}}_{[\kappa, j(\kappa))}$, where $\dot{\mathbb{P}}_{[\alpha,\beta)}$ is (a name for) the restriction of $\mathbb{P}$ to the interval $[\alpha, \beta)$.

We get that $M[G] \models \mathbb{P}_{[\kappa,j(\kappa))}$ is κ^+-closed, since it is an iteration of κ^+-closed forcings with enough support. Now note that $M[G]$ is closed under κ-sequences (from $V[G]$), because M is closed under κ sequences from V and $\mathbb{P}$ is κ-cc. (This latter fact requires some nontrivial analysis of iterations.) And $V[G]$ thinks that $j(\kappa)$ has cardinality κ^+, because $j(\kappa) < (2^\kappa)^+ = \kappa^{++}$ by the assumption that GCH holds at κ. So in $V[G]$ we get that there at most κ^+ many antichains in $\mathbb{P}_{[\kappa,j(\kappa))}$. This implies that in $V[G]$ we can construct $K \subseteq \mathbb{P}_{[\kappa,j(\kappa))}$ which is $M[G]$-generic. (Recall from your poset combinatorics exercises that if $\mathbb{P}$ is λ-closed then you can find a filter meeting λ-many dense sets.) So if we build K then we get that $j''G \subseteq G * K$. So j will lift to $j : V[G] \to M[G * K]$, and this is unique for this choice of generic.

So the question we must face is: how many ways are there to build K? Note that we always have $j''G \subseteq G * K$ no matter how we built K, so that won't be an issue. We build K by enumerating, in $V[G]$, the κ^+ many maximal antichains as $\langle A_i : i < \kappa^+\rangle$. We will use these to build a tree of stronger and stronger conditions, the branches of which will generate different choices of K. We start with $p_\emptyset \in A_0$. Then, because $\mathbb{P}_{[\kappa,j(\kappa))}$ is nontrivial, we can extend $p_\emptyset$ to two incompatible conditions. These conditions can then be extended to meet A_1, giving us $p_{\langle 0\rangle}, p_{\langle 1\rangle} \in A_i$. In general, going from stage i to stage $i+1$ we are faced with conditions p_s where $s \in {}^i 2$. We can extend p_s in two incompatible ways, then extend those conditions to $p_{s^\frown 0}, p_{s^\frown 1} \in A_{i+1}$. And a limit stages ℓ we can find lower bounds in A_ℓ by the κ^+-closure of $\mathbb{P}_{[\kappa,j(\kappa))}$ in $M[G]$.

In the end, we have built a tree of conditions p_s for each $s \in {}^{<\kappa^+}2$, where $s \subseteq t$ implies $p_s \geq p_t$. This is a perfect tree, meaning that each node splits. So, in $V[G]$, it has $2^{\kappa^+} = 2^{2^\kappa}$ many branches. And each of these branches gives a different choice of K, which each give different lifts of j, and hence different normal measures. So we know that in $V[G]$ that there must be at least 2^{2^κ} many normal measures on κ. But that is the maximum possible number, so it must be the precise count. $\square$

This establishes an upper bound for the number of normal measures. What about intermediate values? Can we have fewer, but still more than 1, normal measure on κ?

Theorem 18 (Apter–Cummings–Hamkins)**.** *Let κ be measurable. Then there is a forcing extension in which κ has exactly κ^+ many normal measures.*

Before proving this we need some set-up.

Definition 19. A forcing $\mathbb{P}$ *admits a gap at* δ if $\mathbb{P}$ factors as $\mathbb{Q} * \dot{\mathbb{R}}$ where $\mathbb{Q}$ is nontrivial, $|\mathbb{Q}| < \delta$ and $\mathbf{1}_{\mathbb{Q}} \Vdash \dot{\mathbb{R}}$ is δ-closed.[2]

Theorem 20 (Hamkins)**.** *Let $\mathbb{P}$ admit a gap at δ and $G \subseteq \mathbb{P}$ be V-generic. Suppose $j : V[G] \to M[j(G)]$ is an embedding with critical point $\kappa > \delta$. Then $j \restriction V : V \to M$ is a (definable) class in V.*

We won't have time in this class to prove the gap forcing theorem. The full proof can be found in: Joel David Hamkins, "Gap Forcing", Israel Journal of Mathematics, vol 125, (2001), pp 237–252.

Before we prove the Apter–Cummings–Hamkins result, let me tell you the basic idea. We know, from the Kunen–Paris result, that it's consistent to have 2^{2^κ} many normal measures on κ. So, what if we just collapse 2^{2^κ} to be κ^+? Can this drop down the number of normal measures while not disturbing anything else?

It turns out this doesn't quite work—why should it be that we didn't add new normal measures when collapsing 2^{2^κ}? But with a small tweak this idea will work.

Proof of Apter–Cummings–Hamkins. We may assume κ has 2^{2^κ} many normal measures and that $2^\kappa = \kappa^+$. Consider $\mathbb{P} = \mathrm{Add}(\omega, 1) * \mathrm{Col}(\kappa^+, 2^{2^\kappa})$, with $c * G \subseteq \mathbb{P}$ a V-generic. We will see that in $V[c][G]$ that κ has exactly κ^+ many normal measures.

First, note that because $\mathrm{Add}(\omega, 1)$ is small forcing, that every measure on κ in $V[c]$ is generated by a measure in V. So in $V[c]$ there are 2^{2^κ} many normal measures on κ. Moreover, they remain measures in $V[c][G]$ since $\mathrm{Col}(\kappa^+, 2^{2^\kappa})$ is κ^+-closed, and thus cannot add any subsets to κ. So we get that there are at least κ^+ many normal measures on κ in $V[c][G]$.

For the other direction of the inequality, take $W \in V[c][G]$ a normal measure on κ, and let $j : V[c][G] \to M[c][j(G)]$ be the embedding generated by W. Now note that $\mathbb{P}$ admits a gap at, say, ω_7, exactly by how $\mathbb{P}$ was defined. So by the gap forcing theorem we get that $j \restriction V$ is definable in V. Then $j \restriction V$ lifts to $j^* : V[c] \to M[c]$. Now, define $U \in V[c]$ as $X \in U$ iff $\kappa \in j^*(X)$. Since $V[c]$ and $V[c][G]$ have the same subsets of κ, we get that $U = W$. So we have seen every normal measure on κ in $V[c][G]$ was already in $V[c]$. Thus, $V[c][G]$ has at most $(2^{2^\kappa})^{V[c]} = (\kappa^+)^{V[c][G]}$ many normal measures on κ. $\square$

This result was improved even further, to get that the number of normal measures can be anything you please.

Theorem 21 (S. Friedman–Magidor)**.** *Assume* GCH*. Suppose κ is measurable and let $\mu \leq \kappa^{++}$ be a cardinal. Then there is a cofinality preserving forcing extension in which there are precisely μ many normal measures on κ.*

We won't prove this result, as it is highly nontrivial and requires technology well beyond the scope of this course.

[2]In fact, we only need $\dot{\mathbb{R}}$ to be forced to be δ-*strategically closed*, a weaker notion. But the forcings we work with will have the stronger property.

3. Prikry forcing and the singular cardinals hypothesis

We saw in part 2.2 that the behavior of the continuum function on the regular cardinals is flexible. Any reasonable behavior is consistent with ZFC. But we passed over in silence the issue of what happens with singular cardinals. Now we will return to that issue.

Observe that we may easily arrange so that GCH fails at singular μ. For instance, we can force to make $2^\kappa > \mu^+$ for some regular $\kappa < \mu$, thereby ensuring that $2^\mu \geq 2^\kappa > \mu^+$. A less trivial, and hence more interesting, question is whether μ can be strong limit but GCH fails at μ.

Definition 22. The *singular cardinals hypothesis* SCH asserts that if μ is a singular strong limit then $2^\mu = \mu^+$.

It is immediate that GCH implies SCH, so SCH is consistent with ZFC. Is it consistent that SCH fails?

Theorem 23 (Gitik)**.** *Over* ZFC*, the failure of* SCH *is equiconsistent with the existence of a measurable cardinal κ of Mitchell order κ^{++}.*

So the failure of SCH has large cardinal strength. In this section we will see a version of one direction of this argument. Namely, we will see that if κ is supercompact then there is a forcing extension in which κ is a singular strong limit with $2^\kappa > \kappa^+$.

There are two steps in the argument. The first step is to get a forcing extension in which κ is measurable and $2^\kappa > \kappa^+$. The second step is to force over this model to make κ have countable cofinality while preserving cardinals. We will treat the two parts separately.

We will start with the second part. Our tool will be Prikry forcing.

Definition 24. Let κ be measurable and let U be a normal measure on κ. Then the *Prikry forcing for U* is the poset $\mathbb{P} = \mathbb{P}(U)$ whose conditions p are of the form $p = (s, A)$ where s is a finite subset of κ and $A \in U$. Call s the *stem* of p and A the *upper part* of p. And $\mathbb{P}$ is ordered as $(s, A) \leq (t, B)$ iff $s \supseteq t$ and $\min(s \setminus t) > \max t$ and $s \setminus t \in B$ and $A \subseteq B$. We summarize the conditions on the stem by saying that s_p *end-extends* s_q.

The way to think about Prikry forcing is: we are adding a cofinal ω-sequence to κ. If $p = (s, A)$ is a condition then the stem s is a partial approximation to the sequence and the upper part A is a promise about where that sequence can grow. Stronger conditions grow the stem into the promised land and restrict future promises.

Observation 25. *$\mathbb{P}$ has the κ^+-cc.*

Proof. Because there are only κ many stems and if (s, A) and (s, B) are two conditions with the same stem, then $(s, A \cap B)$ is a condition below each of them. $\square$

Lemma 26 (Prikry lemma)**.** *Let φ be a formula in the forcing language without free variables but possibly with names. Then for any $(s, A) \in \mathbb{P}$ there is $(s, B) \in \mathbb{P}$ so that (s, B) decides φ.*

Because $(s, A \cap B)$ is below both (s, A) and (s, B), this implies that any condition in $\mathbb{P}$ can be extended to a condition which decides φ without changing the stem.

Proof. Let $j : V \to M$ be the embedding derived from U. Fix (s, A). Let $D \subseteq \mathbb{P}$ be the dense open set of all $q \in \mathbb{P}$ so that q decides φ. For each t end-extending s, pick A_t so that $(t, A_t) \in D$, if such exists. Define $A^* = \{\alpha < \kappa : \forall t$ end-extending s we have $\alpha > \max t$ implies $\alpha \in A_t\}$. Let us see that $\kappa \in j(A^*)$. To see this, we want to consider an arbitrary t end-extending s so that $\kappa > \max t$. But in this case, t is a finite subset of κ, so we want to ask whether $\kappa \in j(A_t)$. But this is just true, because $A_t \in U$. Thus, $A^* \in U$. Now note that A^* has the property that if $(t, B) \leq (s, A)$ and $(t, B) \in D$ then $(t, A^* \setminus (\max t + 1)) \in D$, by the construction of A^*.

Let t end-extending s be arbitrary. Partition $A^* \setminus (\max t + 1)$ into three sets:

$$\begin{aligned} B_t^+ &= \{\alpha < \kappa : (t \cup \{\alpha\}, A^* \setminus (\alpha+1) \Vdash \varphi\}, \\ B_t^- &= \{\alpha < \kappa : (t \cup \{\alpha\}, A^* \setminus (\alpha+1) \Vdash \neg\varphi\}, \\ B_t^? &= \{\alpha < \kappa : (t \cup \{\alpha\}, A^* \setminus (\alpha+1) \not\Vdash \varphi\}. \end{aligned}$$

Let B_t be the unique of these three sets which is in U. Set $A^{**} = \{\alpha < \kappa : \forall t$ end-extending s we have $\alpha > \max t$ implies $\alpha \in B_t\}$. Then $A^{**} \in U$. (Exercise: check this, by looking at a certain diagonal intersection.)

Let (u, B) be an extension of (s, A^{**}) so that (u, B) decides φ, where we choose u to have minimal length such that this happens. Suppose toward a contradiction that $u \neq s$. In this case, $u = t^\frown\alpha$ for some t end-extending s. Because (u, B) decides φ and (u, B) is compatible with (s, A^{**}), we get that either $B_t = B_t^+$ or $B_t = B_t^-$, depending upon which way (u, B) decides φ. Now note that every extension of (t, A^{**}) is compatible with a condition of the form $(t^\frown\beta, A^*)$, where $\beta \in B_t$. But then (t, A^{**}) must decide φ, since every condition extending t by lengthening the stem by 1 can be extended to a condition with the same stem which decides φ. But this contradicts the minimality of the length of u. So we must have that $u = s$. So we have found $B \in U$ so that (s, B) decides φ, completing the proof. □

Corollary 27. *$\mathbb{P}$ does not add bounded subsets of κ. In particular, $\mathbb{P}$ does not collapse cardinals $\leq \kappa$.*

Proof. Let $\dot{x}$ be a name for a subset of $\beta < \kappa$. Fix $(s, A_0) \in \mathbb{P}$. Given A_i, let $A_{i+1} \subseteq A_i$ be a set in U so that (s, A_{i+1}) decides whether $\check{\imath} \in \dot{x}$. Such can be found by the Prikry lemma. And at limit stages ℓ set $A_\ell = \bigcap_{i<\ell} A_i$. Then (s, A_β) decides every member $\dot{x}$. So if $G \subseteq \mathbb{P}$ is any V-generic then $\dot{x}_G$ is already in V. □

I want to emphasize what we did here. Clearly, $\mathbb{P}$ is not even ω_1-closed, since it adds a new ω-sequence Previously, we used closure to ensure that we did not add subsets to small cardinals. But that won't work here, so we need to be smarter. This is where the Prikry lemma comes in. It lets us conclude that our forcing doesn't add new subsets to small cardinals, even though it has very little closure. If you don't have the upper parts of conditions then you will not be able to make this conclusion.

Exercise 28. Consider a variant on Prikry forcing where conditions only have the stem s_p. Show that this forcing adds new reals. (Hint: you can define parity for ordinals: α is even iff there is an ordinal β so that $\alpha = 2 \cdot \beta$. Use this to show that this variant on Prikry forcing adds a Cohen real.)

Observation 29. *Let $G \subseteq \mathbb{P}$ be V-generic. Then in $V[G]$ there is a ω-sequence cofinal in κ.*

Proof. Let $\langle \alpha_n : n \in \omega \rangle$ be the sequence enumerating in increasing order the ordinals in the union of the stems of conditions in G. By density, this sequence is cofinal in κ. Done. □

Altogether, what we have proved is the following.

Theorem 30 (Prikry)**.** *Let κ be measurable. Then there is a cardinal-preserving forcing extension in which κ has countable cofinality.* □

In particular, this gives us an example of a forcing which preserves cardinals but does not preserve cofinalities.

This completes part 2 of forcing the negation of SCH. We next want to look at the preparatory forcing we will apply to κ before we hit κ with Prikry forcing. Recall that we want to force GCH to fail at κ.

First, let us see why simply forcing with $\mathrm{Add}(\kappa, \kappa^{++})$ will not do. It may be that GCH holds below κ. But then GCH holds on a set in U, where U is any normal measure on κ. So then we get that GCH holds at κ in M, where $j : V \to M$ is the embedding via U. But M is correct about subsets of κ, so this then implies that $2^\kappa = \kappa^+$. So something must go wrong if we force with $\mathrm{Add}(\kappa, \kappa^{++})$.

Exercise 31. Assume GCH. Show that forcing with $\mathrm{Add}(\kappa, \kappa^{++})$ for κ measurable will preserve the inaccessibility of κ but will destroy the measurability of κ.

The solution is then to first establish the desired reflection of the failure of GCH. That is, before we force GCH to fail at κ we first have to force it fail often below κ. Then we have a hope of making it work.

Assume here that κ is κ^{++}-supercompact. We will assume GCH in the ground model. (If GCH doesn't already hold, we can force it while preserving the partial supercompactness of κ.) Our preparatory forcing will be $\mathbb{A}$, the Easton support product of $\mathrm{Add}(\alpha, \alpha^{++})$ for $\alpha \leq \kappa$ inaccessible. For an inaccessible $\alpha \leq \kappa$, let $\mathbb{A}_\alpha$ be the restriction of $\mathbb{A}$ to the coordinates $< \alpha$. In particular, $\mathbb{A} \cong \mathbb{A}_\kappa * \mathrm{Add}(\kappa, \kappa^{++})$.

Proposition 32. *After forcing with $\mathbb{A}$, we have that κ is still κ^{++}-supercompact and $2^\kappa = \kappa^{++}$.*

Proof. Let $G * H \subseteq \mathbb{A}_\kappa * \mathrm{Add}(\kappa, \kappa^{++})$ be V-generic. That $2^\kappa = \kappa^{++}$ in $V[G][H]$ is clear. The real work is in seeing that κ preserves its large cardinal properties. We saw in section 1 that if a λ-supercompactness embedding lifts, then the lift is λ-supercompactness embedding. So we have only to check that the κ^{++}-supercompactness embedding $j : V \to M$ lifts. Remember the lifting criterion: $j : V \to M$ lifts to $j : V[G * H] \to M[K]$ if $j''G * H \subseteq K$. So we need to find an appropriate M-generic K.

To see what to do, let's analyze $j(\mathbb{A})$. First, note that $\mathbb{A}$ itself is κ-closed, and so $j(\mathbb{A})$ is $j(\kappa)$-closed. Also note that $\mathbb{A}_\kappa$ has size κ and has the κ-cc. Then by elementarity we get that $j(\mathbb{A})$ is the Easton support product of $\mathrm{Add}(\alpha, \alpha^{++})$ at inaccessible $\alpha < j(\kappa)$, where this is defined in M. So we can factor $j(\mathbb{A})$ as $\mathbb{A}_\kappa * \mathrm{Add}(\kappa, \kappa^{++}) * \mathbb{B} * \mathrm{Add}(j(\kappa), j(\kappa)^{++})^M$, where $\mathbb{B}$ is a certain Easton support iteration defined inside M. Note that $M \models \mathbb{B}$ is $j(\kappa)$-closed and that $j(\mathbb{A}_\kappa) = \mathbb{A}_\kappa * \mathrm{Add}(\kappa, \kappa^{++}) * \mathbb{B}$.

What is $|j(\kappa)|^V$? We can calculate this by using the fact that any $\alpha < j(\kappa)$ is of the form $[f]_U$, where U is the normal fine measure giving us the embedding and $f : \mathcal{P}_\kappa \kappa^{++} \to \kappa$. There are

$$\kappa^{(\kappa^{++})^{<\kappa}} = \kappa^{\kappa^{++}} = 2^{\kappa^{++}} = \kappa^{+++}$$

many such functions. So we can conclude that $|j(\kappa)|^V \leq \kappa^{+++}$. Thus $\mathbb{Q}$, as seen from V, is κ^{+++}-closed and has the κ^{+++}-cc. So if we want to build $K \subseteq \mathbb{Q}$ which is $M[G * H]$-generic then we only need to meet κ^{+++} many antichains. This can be done by κ^{+++}-closure. So in $V[G * H]$ we can build such K. It's now easy to see that $j''G \subseteq G * H * K$, since $j''G = G$. So $j : V \to M$ lifts to $j : V[G] \to M[G * H * K]$.

We are not yet done. We still have to lift through to the last coordinate of the iteration $j(\mathbb{A})$. Note that $j''H \subseteq M[G * H]$ has, as seen from V, κ^{++}. So by the closure condition on $M[G * H]$ we get that $j''H \in M[G * H]$. And it is directed by elementarity. Because $V[G * H]$ thinks that $\mathrm{Add}(j(\kappa), j(\kappa)^{++})^M$ is κ^{+++}-closed, this means that we can find a condition p is this poset which is a lower bound for all of $j''H$. We call such a p a *master condition*, because it is a single condition which contains all the information about $j''H$. Again using closure and chain condition facts about the forcing defined in M as viewed in V, we can build a $L \subseteq \mathrm{Add}(j(\kappa), j(\kappa)^{++})^M$ a $M[G * H * K]$-generic. And we can ensure that $p \in L$. This then implies that $j''H \subseteq L$, so we get that $j : V[G] \to M[G * H * K]$ lefts to $j : V[G * H] \to M[G * H * K * L]$. This embedding witnesses that κ remains κ^{++}-supercompact after forcing, and so we are done. $\square$

Putting together this part with the previous part, we get the following.

Theorem 33. *Suppose κ is κ^{++}-supercompact. Then there is a forcing extension in which* SCH *fails.* $\square$

To get this down to the optimal hypothesis (that κ is measurable with Mitchell order κ^{++}) requires a more sophisticated lifting argument. The other direction, that a failure of SCH implies the consistency of a measurable cardinal with high Mitchell order requires other ideas. Rather than being proved by forcing, it is proved by looking at *inner models*. To briefly explain why it cannot be proved by forcing: If we could force to have a measurable cardinal κ in $V[G]$, then in particular we'd have that κ is inaccessible in $V[G]$. But being inaccessible is downward absolute, so κ would have to be inaccessible in V. Now note that if $\neg\mathsf{SCH}$ is consistent then so is $\neg\mathsf{SCH}$ + there are no inaccessibles. This can be proved by a class forcing argument destroying all inaccessibles while preserving the failure of $\neg\mathsf{SCH}$. So it could've been that V had no inaccessibles all along. So this idea could not work.

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