MATH655 LECTURE NOTES: PART 2.2 ACTUALLY DOING THINGS WITH FORCING

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1. How we talk about forcing

In part 2.1, we talked about forcing over a countable transitive model of set theory. The reason for insisting on M to be countable is that that ensured we always had M-generics. But when we're actually working with forcing, it's inconvenient to constantly be talking about some countable M, and relativizing all out statements to M. Instead, we want to imagine ourselves as working inside M and $M[G]$, not caring that there is some larger universe above. So we may as well just imagine that M is V itself. After all, the denizens of M they don't know that they live in a miniverse.

Common practice among set theorists is to talk about forcing over V. So we talk about a forcing extension $V[G]$ of V via a V-generic $G \subseteq \mathbb{P} \in V$.[1] Depending on your philosophical leanings, there's two main ways to make sense of this talk. The first is to treat it as a paraphrase. Rather than saying "such and such is forced by blah blah", it's easier to think in terms of an actual forcing extension of V and talk about what is true in $V[G]$. In this way, you aren't actually saying that there is this generic object G from outside V, which after all is supposed to be the universe of *all* sets. Instead, it's just a convenient framework to reason about the forcing relations.

The other is to not think that there is a universe of all sets. Rather, there is an entire multiverse of universes of sets and forcing is just one way to move between universes. From this perspective, V is indexical. It refers to the current universe, not to some ultimate biggest universe. And then $V[G]$ is just another universe in the multiverse.

You can take whatever perspective you like (and there are others I have not mentioned), but the mathematics is all the same. And no matter which interpretation you take, we get the same theorems about independence, consistency, and so forth. It's just that it's easiest and most convenient to think of $V[G]$ as an actual thing, rather than work directly with the forcing relations. But if a deranged formalist corners you in an ally with a gun and demands that you excise any platonist talk of real universes of sets, you can explain how all of this can be translated into talk about certain statements about the forcing relations being theorems of ZFC, which is a certain set of formulae in a certain formal language.

2. Preservation of cardinals and cofinalities

The capstone of part 2.1 was the argument that forcing preserves ZFC. To do this we had to know about how V and $V[G]$ relate. But our touch was very coarse. If we want to know more about $V[G]$, we'll have to take a finer look. A central question one has to ask when forcing is: what is

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[1] Compare to Kunen's sections IV.3 and IV.4, where he uses the countable transitive model talk. His approach works, but it's clunky to be constantly relativizing statements to M.

absolute between V and $V[G]$? I know that such and such is true in V. When can I conclude that it's also true in $V[G]$? Especially important here are questions about cardinals and cofinalities.

Definition 1. A forcing poset $\mathbb{P}$ is said to *preserve cardinals* if given any V-generic $G \subseteq \mathbb{P}$ we have that κ is a cardinal in V iff κ is a cardinal in $V[G]$.

Definition 2. Let κ be a cardinal (i.e. in V). A forcing poset $\mathbb{P}$ is said to *collapse* κ if $|\kappa|^{V[G]} < \kappa$.

Definition 3. A forcing poset $\mathbb{P}$ is said to *preserve cofinalities* if given any V-generic $G \subseteq \mathbb{P}$ we have that for all ordinals α that $\mathrm{cof}(\alpha)^V = \mathrm{cof}(\alpha)^{V[G]}$.

We can also talk about only preserving some cardinals/cofinalities. For example, $\mathbb{P}$ preserves cardinals $> \lambda$ if every $\kappa > \lambda$ is a cardinal in V iff it is a cardinal in $V[G]$.

Exercise 4. Rework the previous definitions to not mention generics but only refer to the forcing relations.

Recall that being a cardinal is a Π_1-property and hence is downward absolute. So if $V[G] \models \kappa$ is a cardinal then $V \models \kappa$ is a cardinal. So the content of $\mathbb{P}$ preserving cardinals is the other direction. Also note that $\mathrm{cof}(\alpha)^V \geq \mathrm{cof}(\alpha)^{V[G]}$, because any cofinal map $f : \beta \to \alpha$ will also be in $V[G]$ and by absoluteness will still be a cofinal map. But it is conceivable that $\mathrm{cof}(\alpha)^V > \mathrm{cof}(\alpha)^{V[G]}$. For example, if $V[G] \models \omega_1^V$ is countable and $\mathrm{cof}(\alpha)^V = \omega_1$, then $\mathrm{cof}(\alpha)^{V[G]} = \omega$. (And we will later see that it is possible that $V[G]$ thinks ω_1^V is countable.)

Lemma 5. *$\mathbb{P}$ preserves cofinalities iff it preserves regular cardinals. That is, $\mathbb{P}$ preserves cofinalities iff $V \models \kappa$ is regular implies $V[G] \models \kappa$ is regular for any V-generic $G \subseteq \mathbb{P}$.*

Proof. ($\Rightarrow$) This is immediate. If κ is regular in V and $\mathbb{P}$ preserves cofinalities then $\mathrm{cof}(\kappa)^{V[G]} = \mathrm{cof}(\kappa)^V = \kappa$.

($\Leftarrow$) Note that successor ordinals always have cofinality 1 and that being a successor is absolute, so every forcing preserves successor ordinals. So we only have to attend to limit ordinals. Let α be a limit ordinal and suppose $\mathrm{cof}(\alpha)^V = \gamma$. As remarked earlier, $\mathrm{cof}(\alpha)^V \geq \mathrm{cof}(\alpha)^{V[G]}$. For the other direction: Recall that cofinalities of limit ordinals are always regular cardinals. Because $\mathbb{P}$ preserves regular cardinals, γ must be regular in $V[G]$. Supppose toward a contradiction that $\mathrm{cof}(\alpha)^{V[G]} < \mathrm{cof}(\alpha)^V$. Then there is a cofinal map $\delta \to \alpha$ in $V[G]$ for $\delta < \gamma$. In $V[G]$, fix cofinal increasing $d : \delta \to \alpha$ and $g : \gamma \to \alpha$. Now define $f : \delta \to \gamma$ as: $f(i) = \min\{j \in \gamma : d(i) > g(j)\}$. Because d and g are both increasing and cofinal, $f(i)$ is well-defined. But note that f is cofinal in γ, since given any $j \in \gamma$ eventually d gets past $g(j)$. So $\mathrm{cof}(\gamma) \leq \delta$ in $V[G]$, contradicting that γ is regular in $V[G]$. □

Note that this lemma didn't use anything about forcing. All we appealed to were some soft absoluteness facts. The same is true of the following corollary.

Corollary 6. *If $\mathbb{P}$ preserves cofinalities then $\mathbb{P}$ preserves cardinals.*

Proof. By the lemma, $\mathbb{P}$ preserves regular cardinals. But then $\mathbb{P}$ will also preserve limits of regular cardinals. And since every uncountable cardinal is either regular or a limit of regulars, $\mathbb{P}$ will preserve all uncountable cardinals. And note that V and $V[G]$ agree on the cardinals $\leq \omega$. So they agree on all cardinals. □

We can refine this corollary.

Corollary 7. *If $\mathbb{P}$ preserves cofinalities $\geq \lambda$ then $\mathbb{P}$ preserves cardinals $\geq \lambda$.* □

What about the converse to this corollary? If $\mathbb{P}$ preserves cardinals, must it preserve cofinalities? The answer to this is independent of ZFC! If Gödel's axiom of constructibility holds, then the answer is yes. But we will see if there is a measurable cardinal then the answer is no. In fact, we will force to change a measurable cardinal's cofinality to be countable while preserving cardinals. Spooky!

Theorem 8. *If $\mathbb{P}$ has the ccc then $\mathbb{P}$ preserves cofinalities, and hence also cardinals.*

Recall that $\mathbb{P}$ has the ccc if every antichain of $\mathbb{P}$ is countable.

Proof. The key fact used to prove this theorem is the following lemma.

Lemma 9. *Assume $\mathbb{P}$ has the ccc and fix $A, B \in V$. Let $G \subseteq \mathbb{P}$ be V-generic. Suppose $f \in V[G]$ is a function from A to B. Then, there is $F \in V$ a function from A to $\mathcal{P}(B)$ so that for all $a \in A$ we have $f(a) \in F(a)$ and $V \models |F(a)| \leq \aleph_0$.*

In V, we do not have enough information to compute f. But we can narrow down the possibilities to only countably many. This sort of covering lemma is typical in forcing arguments.

Proof. Let $\dot{f}$ be a name for f, so that $\dot{f}_G = f$. Fix $p \in G$ so that $p \Vdash \dot{f} : \check{A} \to \check{B}$. Define $F : A \to B$ as

$$F(a) = \{b \in B : \exists q \leq p \ q \Vdash \dot{f}(\check{a}) = \check{b}\}.$$

Then $F \in V$. Let us see that $f(a) \in F(a)$. Suppose $f(a) = b$. Then there must be $q \in G$ so that $q \Vdash \dot{f}(\check{a}) = \check{b}$. And since G is directed, we have such $q \leq p$. This shows that $b \in F(a)$. And we must see that $|F(a)| \leq \aleph_0$ in V. Toward this end, for each $b \in F(a)$ pick $q_b \leq p$ so that $q_b \Vdash \dot{f}(\check{a}) = \check{b}$. Note that the q_b are pairwise incompatible, since they force contradictory statements. That is, $\{q_b : b \in F(a)\}$ is an antichain. So it must be at most countable, by $\mathbb{P}$ having the ccc. □

Now that we have this lemma, we want to see that forcing with $\mathbb{P}$ which has the ccc preserves cofinalities. We will do this by showing that $\mathbb{P}$ preserves regular cardinals. So take $\kappa \in V$ a regular cardinal. Suppose toward a contradiction that $V[G]$ thinks κ is not regular. Then, in $V[G]$ there is a cofinal map $f : \lambda \to \kappa$ for $\lambda < \kappa$. Now apply the lemma with $A = \lambda$ and $B = \kappa$. Then we get $F : \lambda \to \mathcal{P}(\kappa)$ in V so that $f(\alpha) \in F(\alpha)$ and $|F(\alpha)| \leq \aleph_0$. Because f is cofinal, we get that $g : \lambda \to \kappa$ defined as $g(\alpha) = \sup F(\alpha)$ is also cofinal. But then V thinks κ is singular, a contradiction. □

We can generalize this argument. Recall that $\mathbb{P}$ has the μ-cc if every antichain of $\mathbb{P}$ has cardinality $< \mu$.

Theorem 10. *Suppose $\mathbb{P}$ has the μ-cc where μ is a regular cardinal. Then $\mathbb{P}$ preserves cofinalties $\geq \mu$, and hence also preserves cardinals $\geq \mu$.*

Proof sketch. The first step is to prove a similar lemma.

Lemma 11. *Assume $\mathbb{P}$ has the μ-cc and fix $A, B \in V$. Let $G \subseteq \mathbb{P}$ be V-generic. Suppose $f \in V[G]$ is a function from A to B. Then, there is $F \in V$ a function from A to $\mathcal{P}(B)$ so that for all $a \in A$ we have $f(a) \in F(a)$ and $V \models |F(a)| < \mu$.*

Proof. Exercise! □

Now carry out the same argument as before, but it only applies to cofinalities $\geq \mu$ because that is what we need to get that $\sup F(\alpha) \in \kappa$ for our $F : \lambda \to \kappa$ defined as before. (Exercise: fill in the details.) □

In short, chain conditions ensure the preservation of big cofinalities, and hence of big cardinals.

Exercise 12. Show that if $\mathbb{P}$ is any forcing then $\mathbb{P}$ preserves cofinalities and cardinals $> |\mathbb{P}|$.

We next turn to a property that ensures the preservation of small cardinals and cofinalities. Recall that $\mathbb{P}$ is κ-closed if given any descending sequence $\langle p_i : i < \alpha \rangle$ of conditions in $\mathbb{P}$ where $\alpha < \kappa$, there is a lower bound for the sequence.

Theorem 13. *If $\mathbb{P}$ is κ-closed then $\mathbb{P}$ preserves cardinals and cofinalities $\leq \kappa$.*

Proof. It is enough to show that $\mathbb{P}$ preserves regular cardinals $\leq \kappa$. Let $\lambda \leq \kappa$ be regular, fix $G \subseteq \mathbb{P}$ a V-generic and suppose toward a contradiction that $V[G]$ thinks that λ is not regular. Then in $V[G]$ there is an increasing cofinal map $f : \mu \to \lambda$ for some $\mu < \lambda$. I claim that $f \in V$, contradicting that λ is regular in V.

To see this, fix $p_0 \in G$ so that $p_0 \Vdash \dot{f} : \check{\mu} \to \check{\lambda}$ is increasing and cofinal. Given p_i, for $i < \mu$, let $p_{i+1} \leq p_i$ be a condition in G which decides the value of $\dot{f}(\check{i})$. If $\ell < \mu$ is limit, then by κ-closure let p_ℓ be a lower bound to the sequence $\langle p_i : i < \ell \rangle$ which is in G. Note that while κ-closure doesn't directly say that there is such a condition in G—how could it, when G is outside the universe?—the set $D = \{q \in \mathbb{P} : \forall i < \ell \; q \leq p_i \text{ or } \exists i < \ell \; q \perp p_i\}$ is a dense subset of $\mathbb{P}$. So G must meet it, and the only possibility is that it meets D at a lower bound for the sequence. Finally, let $p_\mu \in G$ be a lower bound for the sequence $\langle p_i : i < \mu \rangle$.

We then get that p_μ decides the value of $\dot{f}(\check{i})$ for every $i < \mu$. And p_μ is in V. So in V we can define f as: $f(i) = \alpha$ iff $p_\mu \Vdash \dot{f}(\check{i}) = \check{\alpha}$. This yields the desired contradiction. □

You can tweak this argument to show a stronger statement.

Exercise 14. Show that if $\mathbb{P}$ is κ-closed then $\mathbb{P}$ doesn't add any new $<\kappa$-sequences. That is, show that if $\alpha < \kappa$ and $\langle x_i : i < \alpha \rangle \in V[G]$, for $G \subseteq \mathbb{P}$ a V-generic, is a sequence of sets from V, then $\langle x_i : i < \alpha \rangle \in V$.

3. The independence of CH

We are finally ready to get our hands on an actual concrete poset!

Definition 15 (Cohen)**.** Let κ, λ be cardinals. Define the poset

$$\mathrm{Add}(\kappa, \lambda) = \{p : p \text{ is a partial function } \kappa \times \lambda \to 2 \text{ with } |p| < \kappa\}.$$

The order for this poset is reverse inclusion. That is, $p \leq q$ iff $p \supseteq q$.

Exercise 16. What is $\mathbf{1}_{\mathrm{Add}(\kappa,\lambda)}$?

Suppose $G \subseteq \mathrm{Add}(\kappa, \lambda)$ is V-generic. We can from G produce a λ-length sequence of subsets of κ, namely by setting $x_i = \bigcup\{p \upharpoonright \kappa \times \{i\} : p \in G\}$. Then x_i is the characteristic function of a subset of κ, and we abusively identify the two. If you think of G as a $\kappa \times \lambda$ grid of 0s and 1s, x_i is the ith column of G. These x_i are known as *Cohen-generics* or *Cohen-generic subsets of* κ. In case $\kappa = \omega$ they are simply called *Cohen reals*. (Reals are either subsets of ω or functions $\omega \to \omega$, depending on context. There is no other use of the term. $\ddot\frown$)

If $\lambda = 1$ we can identify $\mathrm{Add}(\kappa, 1)$ with the collection of partial functions from $\kappa \to 2$, each having cardinality $< \kappa$.

Exercise 17. Suppose $\lambda \leq \kappa$. Show that $\mathrm{Add}(\kappa, 1)$ and $\mathrm{Add}(\kappa, \lambda)$ are forcing equivalent. That is, you need to show the following two things.

- Show that if $G \subseteq \mathrm{Add}(\kappa, 1)$ is V-generic then there is $H \subseteq \mathrm{Add}(\kappa, \lambda)$ in $V[G]$ which is V-generic and $V[G] = V[H]$.
- Show that if $H \subseteq \mathrm{Add}(\kappa, \lambda)$ is V-generic then there is $G \subseteq \mathrm{Add}(\kappa, 1)$ in $V[H]$ which is V-generic and $V[H] = V[G]$.

(Hint: there is a bijection from $\kappa \times 1$ to $\kappa \times \lambda$ under the assumption that $\lambda \leq \kappa$.)

Exercise 18. Let ${}^{<\omega}2$ be the full binary tree of finite binary sequences, ordered by reverse inclusion. And let ${}^{<\omega}\omega$ be the tree of finite sequences of natural numbers, again ordered by reverse inclusion. So the empty sequence is the root and the maximum element, and the tree grows downward. Show that ${}^{<\omega}2$, ${}^{<\omega}\omega$, and $\mathrm{Add}(\omega, 1)$ are forcing equivalent. (Hint: first show that $\mathrm{Add}(\omega, 1)$ embeds densely into ${}^{<\omega}2$.)

Proposition 19. *Suppose κ is regular. Then $\mathrm{Add}(\kappa, \lambda)$ is κ-closed.*

Proof. Suppose $\langle p_i : i < \alpha \rangle$ is a descending sequence of conditions of length $\alpha < \kappa$. That is, $p_i \subseteq p_j$ if $i \geq j$. Now let $p = \bigcup_{i<\alpha} p_i$. Then, because the partial functions are linearly ordered by $\subseteq$, we get that p is a partial function from $\kappa \times \lambda \to 2$. More, each p_i has cardinality $< \kappa$ so the union of all them has cardinality $< \kappa$, by the regularity of κ. Thus, we have seen that $p \in \mathrm{Add}(\kappa, \lambda)$. Now observe that $p \leq p_i$ for each $i < \alpha$. □

Combined with our work from the previous section, we now know that if κ is regular then $\mathrm{Add}(\kappa, \lambda)$ preserves cofinalities and cardinals $\leq \kappa$. In particular, $\mathrm{Add}(\kappa, \lambda)$ doesn't collapse κ.

Next we want to know which big cardinals and cofinalities are preserved by $\mathrm{Add}(\kappa, \lambda)$. For this, we want to know what chain condition it has. First, however, we need a lemma.

Lemma 20 (Delta system lemma)**.** *Consider $\kappa > \lambda$ infinite regular cardinals. Suppose that κ has the property that for all $\mu < \kappa$ we have $\mu^{<\lambda} < \kappa$. Suppose A is a size κ collection of sets of size $< \lambda$. Then there is $B \subseteq A$ of size κ so that B forms a* delta system. *That is, there is a set R so that for all $a \neq b \in B$ we have $a \cap b = R$.*

The special case where $\lambda = \omega$ says that any uncountable collection of finite sets can be thinned out to a delta system of the same cardinality.

Proof. Enumerate A as $\langle a_i : i < \kappa \rangle$. We may assume without loss that each $a_i \subseteq \kappa$. Let $S \subseteq \kappa$ consist of the ordinals $< \kappa$ of cofinality λ. Then S is stationary. (Exercise: give the one-line proof of this.) Now recall some facts about stationary sets and regressive functions, which we proved during the section on measurable cardinals. (There, we were concerned with stationarity with respect to an arbitrary filter, whereas here we are interested in the club filter.) Namely, recall the result known as Fodor's lemma: If $f : E \to \kappa$, where $E \subseteq \kappa$ is stationary, is regressive, then there is $\alpha < \kappa$ so that $f(i) = \alpha$ stationarily often.

Apply Fodor's lemma to the function $f : S \to \kappa$ defined as $f(i) = \sup(i \cap a_i)$. This function is regressive because $|a_i| < \lambda = \mathrm{cof}(i)$. Then we get $T \subseteq S$ stationary so that $f \upharpoonright T$ is constant, say with value α. Consider the club $C = \{\beta < \kappa : \forall i < \beta \ \sup a_i < \beta\}$. Then $T \cap C$ is stationary.

Observe that if $i < j \in T \cap C$, then $\sup A_i < j$ and $\sup(j \cap a_j) = \alpha$. So $\sup(a_i \cap a_j) \leq \alpha$. In other words, $a_i \cap a_j \subseteq \alpha + 1$. Now we use the cardinal arithmetic assumption from the statement of the lemma: each a_i has cardinality $< \lambda$ so this means there are at most $|\alpha + 1|^{<\lambda} < \kappa$ many possibilities for these intersections. Since there are $< \kappa$ many possibilities, this must mean there is a stationary set $W \subseteq T \cap C$ where they all agree. (Otherwise, if each option was nonstationary, then we would have $< \kappa$ many nonstationary subsets unioning up to a stationary set, which is impossible.) Then $B = \{a_i : i \in W\}$ is a delta system. □

Proposition 21. $\operatorname{Add}(\kappa, \lambda)$ *has the* $(2^{<\kappa})^+$*-cc.*

Proof. Let $\mu = (2^{<\kappa})^+$. Suppose for sake of a contradiction that $A \subseteq \mathbb{P} = \operatorname{Add}(\kappa, \lambda)$ is an antichain of cardinality μ. We may without loss only consider the case where κ is regular; otherwise, since $\kappa < \mu$ there must be $\kappa_0 < \kappa$ so that $\{p \in A : |p| = \kappa_0\}$ has size κ, so we can switch to considering that antichain and replace κ with the regular cardinal ${\kappa_0}^+$.

Enumerate A as $\langle p_i : i < \mu \rangle$ and let $S_i = \operatorname{dom} p_i$. By the delta system lemma there is $D \subseteq \mu$ of size μ so that $\{S_i : i \in D\}$ forms a delta system with root R. Note that $2^{|R|} < \mu$, since $|R| < \kappa$. So there must be p_i, p_j with $i \neq j \in D$ so that $p_i \restriction R = p_j \restriction R$. But $R = \operatorname{dom} p_i \cap \operatorname{dom} p_j$, so this means that p_i and p_j are compatible, contradicting that they are elements of the same antichain. □

We are now ready to see that CH can consistently fail.

Theorem 22 (Cohen)**.** $\mathsf{ZFC} + \neg\mathsf{CH}$ *is consistent. (Supposing* ZFC *is consistent.)*

This follows immediately from the following lemma, by taking $\lambda \geq \aleph_2$.

Lemma 23 (Cohen)**.** *Suppose* λ *is uncountable. Let* $G \subseteq \operatorname{Add}(\omega, \lambda)$ *be* V*-generic. Then* $V[G] \models 2^{\aleph_0} \geq \lambda$.

Proof. By the proposition, $\mathbb{P} = \operatorname{Add}(\omega, \lambda)$ has the ccc, because $2^{<\omega} = \omega$. So $\mathbb{P}$ preserves all cardinals. In particular, λ is a cardinal in $V[G]$. Now note that $V[G]$ has at least λ many reals, since it contains the sequence $\langle x_i : i < \lambda \rangle$ of Cohen reals coming from G. □

Exercise 24. Assume $\lambda \geq 2^\kappa$. Show that $\operatorname{Add}(\kappa, \lambda)$ forces $2^\kappa \geq \lambda$.

We would like to know precisely what $V[G]$ thinks the cardinality of the continuum is. First, observe that by Kőnig's theorem it's possible that $V[G] \models 2^{\aleph_0} \neq \lambda$, which happens whenever λ has countable cofinality. So we have to be a bit more careful. We need a bit of technology to more carefully see what forcing does.

Definition 25. Let σ be a $\mathbb{P}$-name. Then a *nice name* for a subset of τ is a name of the form

$$\bigcup \big\{ \{\sigma\} \times A_\sigma : \sigma \in \operatorname{dom}(\tau) \big\},$$

where each $A_\sigma \subseteq \mathbb{P}$ is an antichain.

Proposition 26. *Suppose* $\mathbb{P}$ *is a poset so that* $|\mathbb{P}| = \kappa$*,* $\mathbb{P}$ *has the* λ*-cc. Let* τ *be a* $\mathbb{P}$*-name with* $|\tau| = \mu$*. Assume all these cardinals are infinite. Then there are at most* $(\kappa^{<\lambda})^\mu$ *nice names for subsets of* τ*.*

Proof. By λ-ccness each A_σ has cardinality $< \lambda$. So $\mathbb{P}$ has at most $\kappa^{<\lambda}$ antichains, and hence there are at most $(\kappa^{<\lambda})^\mu$ nice names for subsets of τ. □

Note that if $\lambda = \nu^+$ is a successor cardinal then ths gives the bound of at most $\kappa^{\max(\nu,\mu)}$ many nice names for subsets of τ. In particular, if $\mathbb{P}$ has the ccc then the bound is κ^μ.

Lemma 27. *Every subset of* τ *is given by a nice name. That is, if* σ, τ *are* $\mathbb{P}$*-names then there is a nice name* ρ *so that* $\mathbf{1} \Vdash \sigma \subseteq \tau \Rightarrow \sigma = \rho$*.*

Note that we will use AC to prove this. This is not in general true in the absence of Choice!

Proof. Define $\rho = \bigcup \{ \{\pi\} \times A_\pi : \pi \in \operatorname{dom}(\tau) \}$ where the A_π are chosen so that A_π is an antichain, for each $p \in A_\pi$ we have $p \Vdash \pi \in \sigma$, and A_π is maximal with respect to these two properties. The existence of maximal such antichains is established via a Zorn argument, and once we know they exist we can pick them by AC. So ρ is well-defined.

Now suppose $G \subseteq \mathbb{P}$ is V-generic and suppose $M[G] \models \sigma_G \subseteq \tau_G$. We show that $\sigma_G = \rho_G$. For the $\supseteq$ inclusion: Suppose $a \in \rho_G$. Then, by construction of ρ, there is $\pi \in \mathrm{dom}(\tau)$ so that $a = \pi_G$ and there is $p \in G$ so that $p \Vdash \pi \in \sigma$. So $a = \pi_G \in \sigma_G$, as desired. For the $\subseteq$ inclusion: Suppose toward a contradiction there is $a \in \sigma_G \setminus \rho_G$. Then $a = \pi_G$ for some $\pi \in \mathrm{dom}(\tau)$. Now pick $q \in G$ so that $q \Vdash \pi \in \sigma \wedge \pi \notin \rho$. By the definition of A_π, we have that $q \perp p$ for all $p \in A_\pi$. But then q witnesses that A_π was not actually a maximal antichain, a contradiction. □

Proposition 28. *Suppose $\mathbb{P}$ is a poset with $|\mathbb{P}| = \kappa$, $\mathbb{P}$ has the λ-cc. Let μ be a cardinal and suppose $\delta = (\kappa^{<\lambda})^\mu$. Then, if $G \subseteq \mathbb{P}$ is V-generic we have that $V[G] \models 2^\mu \leq \delta$.*

Proof. Consider the name $\check\lambda$, which clearly has size λ. So there are at most δ many nice names for subsets of $\check\lambda$. Let $\dot f = \{(\mathrm{op}(\check\xi, \rho_\xi), \mathbf{1}) : \xi < \delta\}$, where ρ_ξ is ξth in an enumeration of the nice names for subsets of $\check\lambda$. Then, in $V[G]$ we have that $f = \dot f_G$ is a function with domain δ so that every subset of λ is $f(\xi)$ for some ξ. That is, f witnesses that $2^\lambda \leq \delta$. □

In particular, if $\mathbb{P}$ has the ccc, then in $V[G]$ we have that $2^\mu \leq (\kappa^\lambda)^V$. This then allows us to exactly compute the cardinality of the continuum after forcing with $\mathrm{Add}(\omega, \lambda)$.

Theorem 29. *Suppose $\lambda^{\aleph_0} = \lambda$. Let $G \subseteq \mathrm{Add}(\omega, \lambda)$ be V-generic. Then $V[G] \models 2^{\aleph_0} = \lambda$.*

Proof. Work in $V[G]$. We earlier saw that $2^{\aleph_0} \geq \lambda$. The other direction, using that $|\mathrm{Add}(\omega, \lambda)|^V = \lambda$ and that $(\lambda^\omega)^V = \lambda$, gives that $2^{\aleph_0} \leq \lambda$. So they are equal. □

Exercise 30. Assume GCH. Show that $\mathrm{cof}(\lambda) > \omega$ implies $\lambda^{\aleph_0} = \lambda$.

This exercise shows that, if we start from an assumption of GCH, we can make the continuum equal any cardinal with uncountable cofinality.

Exercise 31. Assume GCH in V. Calculate the cardinality of $2^{\aleph_0}$ in $V[G]$ for $G \subseteq \mathrm{Add}(\omega, \aleph_\omega)$ a V-generic.

Exercise 32. Suppose $\lambda^\kappa = \lambda$ and $\lambda > 2^\kappa$. Let $G \subseteq \mathrm{Add}(\kappa, \lambda)$ be V-generic. Show that $V[G] \models 2^\kappa = \lambda$.

It should be noted that forcing with $\mathrm{Add}(\omega, \lambda)$, while it preserves all cardinals, can be destructive of other properties.

Exercise 33. Show that inaccessibility can be destroyed by forcing. Namely, suppose κ is inaccessible. Show that if $G \subseteq \mathrm{Add}(\omega, \kappa)$ is V-generic then $V[G]$ thinks that κ is not inaccessible.

And if inaccessibility can be destroyed, so can Mahlo-ness, measurability, supercompactness, and so on, as they all imply inaccessibility. A question one might ask: is it possible to, say, destroy measurability while preserving inaccessibility? We will return to these sorts of issues in part 3.

For now, we are looking at the continuum hypothesis. We have just seen that CH can fail, and can fail very badly. Can it succeed?

Proposition 34. $\mathrm{Add}(\omega_1, 1)$ *forces* CH.

Observe that $\mathrm{Add}(\omega_1, 1)$ is ω_1-closed and has the ω_2-cc. So it preserves cardinals.

Proof. Let $G \subseteq \mathrm{Add}(\omega_1, 1)$ be V-generic and let $x = \bigcup G$ be the Cohen-generic subset of ω_1 coming from G. Let's see that every subset of ω is coded into x. By this, I mean that for every $y \subseteq \omega$ there is $\alpha < \omega_1$ so that $n \in y$ iff $\alpha + n \in x$.

Go back to V for now, and fix $y \subseteq \omega$. Observe that given any condition $p \in \mathrm{Add}(\omega_1, 1)$, we can extend p to $p^\frown \chi_y$. Thus, it is dense to code y into the generic. Back in $V[G]$: Since G must meet every dense set in V, we thus have that each $y \subseteq \omega$ from V is coded into x: namely, for each such y there is $\alpha < \omega_1$ so that $n < \omega$ is in y iff $\alpha + n \in x$. And by ω_1-closure of $\mathrm{Add}(\omega_1, 1)$ we get that $V[G]$ doesn't have any new reals. So x codes all the reals. But x only has ω_1 many coding points. So $2^{\aleph_0} = \aleph_1$. □

In fact, this poset forces more than just CH.

Exercise 35 (Hard). Show that $\mathrm{Add}(\omega_1, 1)$ forces $\diamondsuit_{\omega_1}$. (Recall the definition from part 1.2, from the end of section 1.)

The next exercise has you generalize this to show that you can always force GCH to hold at κ.

Exercise 36. Show that $\mathrm{Add}(\kappa^+, 1)$ forces $2^\kappa = \kappa^+$. Show by an example that it is possible for this forcing to also change the continuum function below κ. (Hint: use $\kappa = \omega_1$. You already know that $2^{\aleph_0}$ can be much bigger than $\aleph_2$.)

Let us see another way to force CH, this one being more destructive and collapsing cardinals.

Proposition 37. *Let $\mathbb{P}$ be the poset with conditions p being countable partial one-to-one functions from ω_1 to $\mathcal{P}(\omega)$, ordered by reverse inclusion. Then $\mathbb{P}$ forces* CH.

Proof. First, observe that $\mathbb{P}$ is ω_1-closed, since if $\langle p_i : i \in \omega \rangle$ is a descending chain of conditions then $\bigcup_i p_i$ is a lower bound for the sequence. So $\mathbb{P}$ preserves cardinals and cofinalities $\leq \omega_1$ and $\mathbb{P}$ doesn't add reals. That is, ${\omega_1}^V = {\omega_1}^{V[G]}$ and $\mathcal{P}(\omega)^V = \mathcal{P}(\omega)^{V[G]}$. I will henceforth drop the superscripts when talking about them, since their value doesn't depend upon in which of V or $V[G]$ we do the calculation.

I claim that if $G \subseteq \mathbb{P}$ is V-generic then $f = \bigcup G$ is a bijection from ω_1 to $\mathcal{P}(\omega)$. First, that f is a function is immediate since each element of G is a function and all functions in G agree on the intersection of their domain. Suppose that f were not one-to-one. Then there would be $\alpha, \beta \in \omega_1$ so that $f(\alpha) = f(\beta)$. But then the functions $p_\alpha = \{(\alpha, f(\alpha))\}$ and $p_\beta = \{(\beta, f(\beta))\}$ are conditions in G. But they are incomaptible, contradicting that G is directed. Finally, we want to see that f is onto $\mathcal{P}(\omega)$. This follows by a density argument. Namely, fix $x \in \mathcal{P}(\omega)$. Then given any $p \in \mathbb{P}$ either $x \in \operatorname{ran} p$ or else we can extend p to $p' = p \cup \{(\sup(\operatorname{dom} p) + 17, x)\}$. This shows that there are densely many p with $x \in \operatorname{ran} p$. So by density G must contain such a p and thus $x \in \operatorname{ran} f$.

Altogether, we have seen that $V[G]$ has a bijection from ω_1 to $\mathcal{P}(\omega)$. So $V[G] \models \mathsf{CH}$. □

Exercise 38. Analyze this forcing in the case that V already satisfies CH. Also, show that in case $V \not\models \mathsf{CH}$ that $\mathbb{P}$ collapses cardinals. (Which cardinals?)

The forcing from the above proposition is a specific case of a more general class of forcing notions.

Definition 39. Let $\kappa \leq \lambda$ be cardinals. Then the *collapse forcing* to collapse λ to κ is

$$\mathrm{Col}(\kappa, \lambda) = \{p : p \text{ is a partial function } \kappa \to \lambda \text{ and } |p| < \kappa\}.$$

Conditions are ordered by reverse inclusion. That is, $p \leq q$ iff $p \supseteq q$.

Exercise 40. Show that $\mathrm{Col}(\kappa, \lambda)$ is κ-closed. Show that $\mathrm{Col}(\kappa, \lambda)$ has the $(\lambda^{<\kappa})^+$-cc. Show that if $G \subseteq \mathrm{Col}(\kappa, \lambda)$ is V-generic then $V[G] \models |\lambda| = \kappa$.

Exercise 41. Show that $\mathrm{Col}(\kappa^+, 2^\kappa)$ preserves cofinalities and cardinals $> 2^\kappa$ and that it forces the GCH to hold at κ.

Combining the results and exercises in this chapter, we have a good amount of control the pattern of the continuum function $\kappa \mapsto 2^\kappa$. For example, we can force $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_2$, $2^{\aleph_2} = \aleph_4$, and $2^{\aleph_4} = \aleph_{16}$. Namely, we first force one of these, then force again to get the next, and so on finitely many times.

But this is a set theory class. Once we know we can do something finitely many times, we would like to know whether we can do it infinitely many times. The answer is yes, but we need to be a little careful, as the following exercise illustrates.

Exercise 42. Let M be a countable transitive model of set theory. Show that there is an ω-length sequence

$$M = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n \subseteq \cdots$$

so that each M_{n+1} is a forcing extension of M_n by a poset in M_n but $\bigcup_n M_n$ is not a model of ZFC. (Hint: because M is countable, externally to M we can find a cofinal sequence $\langle \kappa_n : n \in \omega \rangle$ in the cardinals of M. Consider the forcings $\mathrm{Add}(\omega, \kappa_n)$, as defined in the M_n's.)

So we can't just willy-nilly take limits. We need to be more clever than that, a topic we now turn to.

4. Product forcing

The trouble in the exercise at the end of the previous section is that the sequence $\langle \kappa_n \rangle$ was not definable over M. As a countable transitive model of set theory, M is not closed under ω-sequences. By looking at an ω-sequence which is not in M, but each of its elements is in M, we can construct an ω-length iteration of forcing extensions whose union isn't a model of ZFC, let alone a forcing extension of M.

The fix to this problem is to ensure that the iteration of posets is itself visible to M. Of course, intermediate stages of the iteration of forcing extensions are themselves extensions of M, and so may contain new posets. So M may not have those posets. But M will have names for them. This will give us a way to define iterations internally to M.

First, however, we will look at products, where all the multiplicands are in the ground model. These are easier to get one's hands on, so they make for a good first look. I remind you that we speak of forcing over V, and will forget the talk of countable transitive models from the previous two paragraphs.

Definition 43. Let $\mathbb{P}$ and $\mathbb{Q}$ be posets. Then $\mathbb{P} \times \mathbb{Q}$ is the *product poset*, defined to have domain $\mathbb{P} \times \mathbb{Q}$ (i.e. the cartesian product of their domains) with the relation given by $(p, q) \le (p', q')$ iff $p \le_{\mathbb{P}} p'$ and $q \le_{\mathbb{Q}} q'$.

Exercise 44. Show that $\mathbb{P} \times \mathbb{Q}$ is actually a poset, and $\mathbf{1}_{\mathbb{P}\times\mathbb{Q}} = (\mathbf{1}_{\mathbb{P}}, \mathbf{1}_{\mathbb{Q}})$.

Proposition 45. *Suppose $K \subseteq \mathbb{P} \times \mathbb{Q}$ is V-generic. Set $G = \{p \in \mathbb{P} : (p, q) \in K$ for some $q\}$ and $H = \{q \in \mathbb{Q} : (p, q) \in K$ for some $p\}$. Then $G \subseteq \mathbb{P}$ and $H \subseteq \mathbb{Q}$ are V-generic and $K = G \times H$.*

Proof. Let $D \subseteq \mathbb{P}$ be dense. Then $D \times \mathbb{Q}$ is a dense subset of $\mathbb{P} \times \mathbb{Q}$. So K meets $D \times \mathbb{Q}$, so G meets D. Since D was arbitrary, G is generic. Similarly one can show that H is generic. And one sees that $K \subseteq G \times H$ by considering $(p, q) \in K$ and immediately getting $p \in G$ and $q \in H$. To see $G \times H \subseteq K$ take $p \in G$ and $q \in H$. Then $(p, \mathbf{1}_{\mathbb{Q}}), (\mathbf{1}_{\mathbb{P}}, q) \in G$. Since K is directed, take (p', q') below both of these conditions. Then $(p', q') \le (p, q)$, so by upward closure of K we get $(p, q) \in K$. $\square$

This is a special case of a more general phenomenon. Say that $e : \mathbb{P} \to \mathbb{Q}$ is a *complete* embedding if

(1) $e(\mathbf{1}_\mathbb{P}) = \mathbf{1}_\mathbb{Q}$;
(2) e preserves order. That is, $p \leq p'$ implies $e(p) \leq e(p')$;
(3) e preserves incompatibility. That is, $p \perp p'$ implies $e(p) \perp e(p')$; and
(4) e preserves maximal antichains. That is, if $A \subseteq \mathbb{P}$ is a maximal antichain in $\mathbb{P}$ then $e''A$ is a maximal antichain in $\mathbb{Q}$.

Exercise 46. Show that the canonical embedding of $\mathbb{P}$ into $\mathbb{P} \times \mathbb{Q}$ is a complete embedding.

Exercise 47. Show that if $e : \mathbb{P} \to \mathbb{Q}$ is a complete embedding then any $H \subseteq \mathbb{Q}$ generic determines a generic subset of $\mathbb{P}$, namely $G = e^{-1}(H) = \{p \in P : e(p) \in H\}$. Conclude that $V[G] \subseteq V[H]$.

Theorem 48. *Let $\mathbb{P}, \mathbb{Q}$ be posets, and suppose $G \subseteq \mathbb{P}$ and $H \subseteq \mathbb{Q}$, where G and H may come from outside V. Then the following are equivalent.*

(1) *$G \times H \subseteq \mathbb{P} \times \mathbb{Q}$ is V-generic.*
(2) *G is V-generic and H is $V[G]$-generic.*
(3) *H is V-generic and G is $V[H]$-generic.*

More, if these hold, then $V[G \times H] = V[G][H] = V[H][G]$.

Proof. ($1 \Rightarrow 2$) We already saw that G is V-generic. Toward seeing H is $V[G]$-generic, fix $D \subseteq \mathbb{Q}$ in $V[G]$. Let $\dot{D} \in V^\mathbb{P}$ be a name for D and fix $p \in G$ so that $p \Vdash \dot{D}$ is a dense subset of $\check{\mathbb{Q}}$. Set $E = \{(p', q') \in \mathbb{P} \times \mathbb{Q} : p' \leq p \text{ and } p' \Vdash q' \in \dot{D}\}$. Then $E \in V$. Let us see that E is dense below $(p, \mathbf{1})$. To see this, fix $(p_0, q_0) \leq (p, \mathbf{1})$. Then $p_0 \Vdash \dot{D}$ is dense in $\check{\mathbb{Q}}$. Thus $p_0 \Vdash \exists r \in \check{\mathbb{Q}}\ r \leq q_0$ and $r \in \dot{D}$. So we get $p' \leq p_0$ and $q' \leq q_0$ so that $p' \Vdash q' \in \dot{D}$. That is, $(p', q') \in E$, as desired.

Thus, since $G \times H$ is generic, we can take $(P, q) \in (G \times H) \cap E$. Then $q \in \dot{D}_G = D$. So $q \in H \cap D$.

($1 \Rightarrow 3$) Proved similarly.

($2 \Rightarrow 1$) That $G \times H$ is a filter is because it is a product of filters. (Exercise: check this!) So we have only to see it is generic. Fix $D \subseteq \mathbb{P} \times \mathbb{Q}$ in V a dense set. Set $D' = \{q \in \mathbb{Q} : \exists p \in G\ (p, q) \in D\}$, working in $V[G]$. Note that if H meets D' then $G \times H$ meets D. Now fix $q_0 \in \mathbb{Q}$. Then $\{p \in \mathbb{P} : \exists q \leq q_0\ (p, q) \in D\}$ in V is dense so we can pick $p \in G$ which meets this dense set. But then $(p, q) \in D$ for some $q \leq q_0$ and so $q \in D^*$, as desired.

($3 \Rightarrow 1$) Proved similarly. □

We want to extend this analysis from two multiplicands to infinitely many multiplicands.

Definition 49. Let I be an index set and let $\mathbb{P}_i$ be posets for each $i \in I$. Then the *full support product* of the $\mathbb{P}_i$ is the poset $\prod_{i \in I} \mathbb{P}_i$ whose elements are are functions p with domain I so that for each $i \in I$ we have that $p(i) \in \mathbb{P}_i$. The order relation for the product is done coordinate-wise: $p \leq q$ iff for all $i \in I$ we have $p(i) \leq_{\mathbb{P}_i} q(i)$.

Full support here means that conditions in the product can contain nontrivial information in each multiplicand. We also look at other notions of support which limit how many multiplicand posets each condition can give nontrivial information about. It must be emphasized that the choice of support can be very important. Different choices can affect whether a forcing adds reals, which cardinals are preserved and so on.

Here is one example of another notion of support we might use.

Definition 50. Let I and $\mathbb{P}_i$ be as before. Then the *finite support product* of the $\mathbb{P}_i$ is the subposet of the full support product where we only allow conditions p so that $p(i) \neq \mathbf{1}_{\mathbb{P}_i}$ finitely often. In what is perhaps an abuse of notation, we also denote this $\prod_{i \in I} \mathbb{P}_i$.

Another example, which we will use in forcings to change the continuum pattern, is due to Easton.

Definition 51. Suppose I is a collection of regular cardinals, possibly a proper class. For each $\alpha \in I$ suppose $\mathbb{P}_\alpha$ is a poset. Then the *Easton support product* of the $\mathbb{P}_\alpha$ is collection of functions p with $\operatorname{dom} p \subseteq I$ a set and so that if κ is weakly inaccessible then $p(\alpha) \neq \mathbf{1}_{\mathbb{P}_\alpha}$ for $< \kappa$ many $\alpha < \kappa$. In other words, below a weakly inaccessible cardinal $p(\alpha) \neq \mathbf{1}_{\mathbb{P}_\alpha}$ only boundedly often.

Conditions are ordered first by extension of their domain and then coordinate-wise. That is, $p \leq q$ iff $\operatorname{dom}(p) \supseteq \operatorname{dom}(q)$ and for all $i \in \operatorname{dom}(q)$ we have $p(i) \leq q(i)$.

The reason to not require each p to have $\operatorname{dom} p = I$ is that we will want to later apply this in the case where I is a proper class. The idea is, if a coordinate is not in the domain of p then we think of that no information as basically being a $\mathbf{1}$ in the coordinate.

Before we can see why this wacky support is useful, we need to see the forcings we will use.

Definition 52. Call a function E a *Easton index function* if $\operatorname{dom}(E)$ is a set of regular cardinals and E satisfies the following two conditions.

(1) For $\alpha \in \operatorname{dom}(E)$ we have $\operatorname{cof}(E(\alpha)) > \alpha$; and
(2) For $\alpha < \beta \in \operatorname{dom}(E)$ we have $E(\alpha) \leq E(\beta)$.

The intuition is: E represents a possible behavior for the continuum function $\kappa \mapsto 2^\kappa$ on a set-sized domain. (1) is required by König's theorem while (2) is the obvious monotonicity property of the continuum function.

Definition 53. Let E be an Easton index function with domain I. Then the *Easton poset* $\mathbb{P}(E) = \prod_{\alpha\in I} \operatorname{Add}(\alpha, E(\alpha))$ is the product with Easton support.

Lemma 54. *Suppose E is an Easton index function with* $\operatorname{dom}(E) = I \subseteq \lambda^+$*, where λ is a regular cardinal so that $2^{<\lambda} = \lambda$. Then $\mathbb{P}(E)$ has the λ^+-cc.*

Proof. Suppose $A = \{p_i : i < \lambda^+\} \subseteq \mathbb{P}(E)$, where the p_i are distinct. We want to see that A is not an antichain. First, note that we may assume each p_i has domain I; otherwise extend p_i by putting $\mathbf{1}$ in each new coordinate. This will not affect whether p_i and p_j are compatible. For each $i < \lambda^+$, set $D_i = \bigcup\{\{\alpha\} \times \operatorname{dom} p_i(\alpha) : \alpha \in I\}$. Observe that $|D_i| < \lambda$. This is trivial in case λ is not weakly inaccessible, as in that case there are $< \lambda$ many regular cardinals $< \lambda$. In case λ is weakly inaccessible, the Easton support condition forces that $p_i(\alpha) \neq \mathbf{1}$, and thus $\operatorname{dom} p_i(\alpha) \neq \emptyset$, can only happen $< \lambda$ often. Now apply the delta system lemma, using that $2^{<\lambda} = \lambda$, to get $J \subseteq I$ of cardinality $|I| = \lambda^+$ so that there is a root R with $D_i \cap D_j = R$ for $i \neq j \in J$. Because $2^{|R|} \leq 2^{<\lambda} = \lambda$ there must be $i \neq j \in J$ so that $p_i(\alpha)(s) = p_j(\alpha)(s)$ for all $(\alpha, s) \in R$. So then p_i and p_j are compatible, so A is not an antichain. □

This lemma also applies when $\operatorname{dom} E \not\subseteq \lambda^+$, by splitting $\mathbb{P}(E)$ into pieces.

Definition 55. Let E be an Easton index function and λ be a cardinal. Then set $E^+_\lambda = E \restriction \{\alpha : \alpha > \lambda\}$ and set $E^-_\lambda = E \restriction \{\alpha : \alpha \leq \lambda\}$.

Observe that $\mathbb{P}(E) \cong \mathbb{P}(E^-_\lambda) \times \mathbb{P}(E^+_\lambda)$ for any λ.

Corollary 56. *Assume* GCH*. Then if E is any Easton index function we have that $\mathbb{P}(E)$ preserves cofinalities and cardinals.*

Proof. Toward a contradiction assume θ is regular in V but θ is not regular in $V[G]$, where $G \subseteq \mathbb{P}(E)$ is V-generic. Let $\lambda = \operatorname{cof}(\theta)^{V[G]} < \theta$. Because being regular is downward absolute, λ is regular in V. Let $f : \lambda \to \theta$ in $V[G]$ be cofinal and increasing.

Now split $\mathbb{P}(E)$ into $\mathbb{P}(E^-_\lambda) \times \mathbb{P}(E^+_\lambda)$, and correspondingly split G into $G^- \times G^+$. Then $V[G] = V[G^-][G^+]$. Observe that $\mathbb{P}(E^+_\lambda)$ is λ^+-closed in V, and so it doesn't add λ-sequences. So $V[G^+]$ thinks that $2^{<\lambda} = \lambda$ and thus $\mathbb{P}(E^-_\lambda)^{V[G]} = \mathbb{P}(E^-_\lambda)^V$. Now apply the lemma inside $V[G^+]$ to get that $\mathbb{P}(E^-_\lambda)^{V[G]}$ is λ^+-cc inside $V[G^+]$. Thus, we can cover f with a function $F : \lambda \to \mathcal{P}(\theta)$ with $F \in V[G^+]$ so that $f(i) \in F(i)$ and $|F(i)| \leq \lambda$ for all $i < \lambda$. Again using the λ^+-closue of $\mathbb{P}(E^+_\lambda)$, we get that $F \in V$. But then in V we have that $\bigcup_{i<\lambda} F(i)$ is a cofinal subset of θ which has cardinality λ. This contradicts that θ is regular in V. □

So why is the Easton support condition the way it is? Because it's exactly what's needed to prove $\mathbb{P}(E)$ preserves cofinalities and cardinals.

Remark 57. For the lemma and its corollary, we don't need the cofinality and monotonicity restrictions on E, which were used nowhere in the proofs. They work for any Easton support product of Cohen forcing on regular cardinals.

Let us now see that we can control the continuum function on set-sized pieces. In a later section we will see how to control the continuum function globally.

Theorem 58 (Easton)**.** *Assume* GCH. *Let E be an Easton index function and suppose $G \subseteq \mathbb{P}(E)$ is V-generic. Then in $V[G]$ for all $\kappa \in \operatorname{dom} E$ we have $2^\kappa = E(\kappa)$.*

In brief, Easton's theorem shows that (on set-sized pieces) the only restraint ZFC puts on the continuum function is Kőnig's theorem and that it must be monotonic.

Proof. We have seen that $\mathbb{P}(E)$ preserves cofinalities and cardinals. Now fix $\kappa \in \operatorname{dom} E$. That $2^\kappa \geq E(\kappa)$ is proved as in the case for forcing with $\operatorname{Add}(\kappa, E(\kappa))$. The point is, just the multiplicand at κ is enough to add all the subsets of κ. For the other direction, that $2^\kappa \leq E(\kappa)$ we factor $\mathbb{P}(E)$ and count nice names. Namely, factor $\mathbb{P}(E)$ as $\mathbb{P}(E^-_\kappa) \times \mathbb{P}(E^+_\kappa)$, and correspondingly split the generic G into $G^- \times G^+$. Then $|\mathbb{P}(E^-_\kappa)| = E(\kappa)$ and $\mathbb{P}(E^-_\kappa)$ has the κ^+-cc. So there are $E(\kappa)$ many nice $\mathbb{P}(E^-_\kappa)$-names for subsets of κ. And forcing with $\mathbb{P}(E^+_\kappa)$ doesn't add κ-sequences. So in $V[G]$ we have $2^\kappa \leq E(\kappa)$. □

I want to mention a particular case of Easton's theorem, which many find striking the first time they see it.

Corollary 59. *Assume* GCH. *Fix a cardinal κ so that every singular cardinal $< \kappa$ is below a regular cardinal $< \kappa$. (For example, this works if κ is a successor of a successor.) It is consistent with* ZFC *to have $2^\lambda = \kappa$ for all $\lambda < \kappa$.*

Proof. Force with $\mathbb{P}(E)$, where is the Easton index function with $E(\lambda) = \kappa$ for all regular $\lambda < \kappa$. Then in the forcing extension $2^\lambda = \kappa$ for all regular $\lambda < \kappa$. And the same holds for singular $\lambda < \kappa$ by monotonicity. □

Let us finish off this section by looking at an important forcing defined using products. Recall that $\operatorname{Col}(\kappa, \lambda)$, for $\kappa \leq \lambda$, is the forcing to collapse λ to have cardinality λ.

Definition 60. Let $\kappa < \lambda$, where λ is inaccessible. Then the *Lévy collapse* $\operatorname{Col}(\kappa, <\lambda)$ is the $<\lambda$-support product of $\operatorname{Col}(\kappa, \mu)$ for $\kappa < \mu < \lambda$. Here, $<\lambda$-support means that each condition p has $p(i) \neq \mathbf{1}$ on a set of size $<\lambda$.

The point is that the Lévy collapse will "softly" collapse the inaccessible λ to be κ^+.

Exercise 61. Fix $\kappa < \lambda$, where λ is inaccessible. Show that $\mathrm{Col}(\kappa, < \lambda)$ is κ-closed and has the λ-cc. Show that if $G \subseteq \mathrm{Col}(\kappa, < \lambda)$ is V-generic then $V[G]$ thinks $\lambda = \kappa^+$.

5. Iterated forcing

The original motivation I gave for product forcing came from iterating forcing extensions. We had a chain of forcing extensions

$$V \subseteq V[G_0] \subseteq V[G_0][G_1] \subseteq V[G_0][G_1][G_2] \subseteq \cdots \subseteq V[G_0][G_1][G_2]\cdots[G_n]$$

and we had asked how to extend this from a chain of length $< \omega$ to one of length $\geq \omega + 1$. We earlier focused on the case where $G_i \subseteq \mathbb{P}_i \in V$, that is where all the posets were in the ground model. But we are also interested in the more general case, where $G_1 \subseteq \mathbb{P}_1 \in V[G_0]$, and so forth. It is this more general case, known as *iterated forcing* to which we know turn. We will start with the simplest case, namely two-step iterations.

Recall that every set in $V[G]$, where $G \subseteq \mathbb{P}$ is V-generic, is given by a $\mathbb{P}$-name in V. This includes any forcing posets in $V[G]$. So if we want to force with $\mathbb{P} \in V$ followed by $\mathbb{Q} \in V[G]$, then in V we want to look at certain $\mathbb{P}$-names.

Definition 62. Let $\mathbb{P}$ be a poset and let $\dot{\mathbb{Q}}$ be a name so that $\mathbf{1}_{\mathbb{P}} \Vdash \dot{\mathbb{Q}}$ is a poset. Then the two-step iteration $\mathbb{P} * \dot{\mathbb{Q}}$ has domain consisting of pairs $(p, \dot{q})$ with $p \in \mathbb{P}$ and $\dot{q} \in \mathrm{dom}(\dot{\mathbb{Q}})$ so that $p \Vdash \dot{q} \in \dot{\mathbb{Q}}$. The order relation on $\mathbb{P} * \dot{\mathbb{Q}}$ is defined as $(p_0, \dot{q}_0) \leq (p_1, \dot{q}_1)$ iff $p_0 \leq p_1$ and $p_0 \Vdash \dot{q}_1 \leq \dot{q}_0$.

There is a detail which needs to be cleared up. While $\dot{\mathbb{Q}}$ may be forced to be a poset, it may not be so clear what $\mathbf{1}_{\mathbb{Q}}$ should be. But in $V[G]$ we may freely replace $\mathbb{Q}$ by an isomorphic copy and obtain the same forcing extensions. In particular, we may arrange so that $\mathbf{1}_{\mathbb{Q}} \in V$, say that $\mathbf{1}_{\mathbb{Q}} = 0$. So we will tacitly assume that $\mathbf{1}_{\mathbb{Q}}$ is a distinguished element of the ground model and that $\mathbf{1}_{\mathbb{P}}$ decides the value of $\dot{\mathbf{1}}_{\mathbb{Q}}$.

Exercise 63. Check that $\mathbb{P} * \dot{\mathbb{Q}}$ is a poset.

Exercise 64. Check that if $\mathbb{Q} \in V$ then $\mathbb{P} * \check{\mathbb{Q}}$ is isomorphic to $\mathbb{P} \times \mathbb{Q}$.

Note that unlike with products, where we can freely swap the order, we cannot do so for iterations. Indeed, $\dot{\mathbb{Q}} * \mathbb{P}$ is undefined nonsense.

Here are some basic facts about two-step iterations. Here, let $e : \mathbb{P} \to \mathbb{P} * \dot{\mathbb{Q}}$ be the embedding $e(p) = (p, \mathbf{1}_{\mathbb{Q}})$.

Fact 65. *The following are true.*

1. *e is actually an embedding of posets.*
2. *If $p \perp p'$ then for all $(p, \dot{q}), (p', \dot{q}') \in \mathbb{P} * \dot{\mathbb{Q}}$ we have $(p, \dot{q}) \perp (p', \dot{q}')$.*
3. *For all $(p, \dot{q}) \in \mathbb{P}$ and all $p' \in \mathbb{P}$ we have $p \perp p'$ iff $(p, \dot{q}) \perp (p', \mathbf{1})$.*
4. *$p \perp p'$ iff $e(p) \perp e(p')$.*
5. *e is a complete embedding. (That is, it preserves maximal antichains.)*

Proof. These are either immediate or one-line arguments. I leave them to you. □

Definition 66. Suppose $G \subseteq \mathbb{P}$ is V-generic, and let $H \subseteq \dot{\mathbb{Q}}_G$. Then $G * H = \{(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}} : p \in G \text{ and } \dot{q}_G \in H\}$.

Theorem 67. *Suppose $K \subseteq \mathbb{P} * \dot{\mathbb{Q}}$ is V-generic. Let $G = e^{-1}(K) = \{p \in \mathbb{P} : (p, \mathbf{1}) \in K\}$. Set $H = \{\dot{q}_G : \dot{q} \in \operatorname{dom}(\dot{\mathbb{Q}})$ and $\exists p\ (p, \dot{q}) \in K\}$. Then $G \subseteq \mathbb{P}$ is V-generic, $H \subseteq \dot{\mathbb{Q}}_G$ is $V[G]$-generic, and $V[K] = V[G][H]$.*

Proof. First, that G is V-generic is because e is a complete embedding. Next, we must see that H is a filter and meets every dense set in $V[G]$. That $\mathbf{1}_{\mathbb{Q}} \in H$ is because $(\mathbf{1}_{\mathbb{P}}, \dot{\mathbf{1}}_{\mathbb{Q}}) \in K$, and by our assumption, made without loss, that $\mathbf{1}_{\mathbb{P}}$ decides the value of $\mathbf{1}_{\mathbb{Q}}$. Too see that H is directed, take $q, q' \in H$. Then there are $p, p' \in G$ so that $(p, \dot{q}), (p', \dot{q}') \in K$, where $q = \dot{q}_G$ and $q' = \dot{q}'_G$. Because K is a filter, this means there is $(p'', \dot{q}'')$ below both of these conditions. That is, $p'' \Vdash \dot{q}'' \leq \dot{q}, \dot{q}'$. And since $p'' \in G$, we get that $q'' = \dot{q}''_G \leq q', q$. But $q'' \in H$, by definition.

Now we see that H is closed upward. Take $q = \dot{q}_G \in H$. We have $p \in G$ so that $(p, \dot{q}) \in K$. Take any $q' \geq q$ in $\mathbb{Q}$. This happens iff there is $p' \in G$ so that $p' \Vdash \dot{q}' \geq \dot{q}$. By directedness of K, there is $(p'', \dot{q}'') \leq (p', \mathbf{1}), (p, \dot{q})$ which is in K. In particular, $p'' \Vdash \dot{q}'' \leq \dot{q}$. But also $p'' \leq p'$, so $p'' \Vdash \dot{q} \leq \dot{q}'$. So we get that $(p'', \dot{q}'') \leq (p', \dot{q}')$, so by upward closure of K we get that $(p', \dot{q}') \in K$. But then $q' \in H$, as desired.

Next we check genericity. Take $D = \dot{D}_G \subseteq \mathbb{Q}$ in $V[G]$ dense. Take $p_0 \in G$ so that $p_0 \Vdash \dot{D}$ is dense. Set $D' = \{(p, q) \in \mathbb{P} * \dot{\mathbb{Q}} : p \leq p_0 \text{ and } p \Vdash q \in \dot{D}\}$. Similar to the product case, you can see that D' is dense below $(p_0, \mathbf{1})$. So there is $(p, \dot{q}) \in K \cap D'$. But then $q = \dot{q}_G \in D$ and $q \in H$. So $H \cap D \neq \emptyset$, as desired.

Next we check that $K = G * H$. For the $\subseteq$ inclusion: if $(p, \dot{q}) \in K$ then $p \in G$ so $\dot{q}_G \in H$ so $(p, \dot{q}) \in G * H$. For the $\supseteq$ inclusion: if $(p, \dot{q}) \in G * H$ then $p \in G$ and $\dot{q}_G \in H$. So $(p, \mathbf{1}) \in K$ and $(p', \dot{q}) \in K$ for some p'. But then there is $(p'', \dot{q}') \leq (p, \mathbf{1}), (p', \dot{q})$ which is in K. So then $p'' \leq p$ and $p'' \Vdash \dot{q}' \leq \dot{q}$. So $(p, \dot{q}) \in K$.

Finally, to see that $V[K] = V[G][H]$ observe that $K = G * H \in V[G][H]$ so $V[K] \subseteq V[G][H]$. For the other inclusion, $V[G][H] \subseteq V[K]$ because $G, H \in V[K]$. □

There is a converse to this result.

Exercise 68. Suppose $G \subseteq \mathbb{P}$ is V-generic and $H \subseteq \dot{\mathbb{Q}}_G$ is $V[G]$-generic. Set $K = H * G$. Then $K \subseteq \mathbb{P} * \dot{\mathbb{Q}}$ is V-generic.

Having seen two-step iterations, let us now see how to extend this transfinitely. You may want to sit down first.

Definition 69. Let α be an ordinal. Then an *α-stage iterated forcing* is a pair of sequences $\langle \mathbb{P}_\xi : \xi \leq \alpha \rangle$ and $\left\langle \dot{\mathbb{Q}}_\xi : \xi < \alpha \right\rangle$ so that the following.

(1) Each $\mathbb{P}_\xi$ is a forcing poset, with its $\leq$ and $\mathbf{1}$.
(2) Each $\dot{\mathbb{Q}}_\xi$ is a $\mathbb{P}_\xi$-name for a forcing poset.
(3) Each $p \in \mathbb{P}_\xi$ is a sequence of the form $\langle \dot{q}_i : i < \xi \rangle$ where each $\dot{q}_i \in \operatorname{dom}(\dot{\mathbb{Q}}_i)$.
(4) If $\xi < \eta$ and $p \in \mathbb{P}_\eta$ then $p \upharpoonright \xi \in \mathbb{P}_\xi$.
(5) If $\xi < \eta$ and $p \in \mathbb{P}_\xi$ and p' is an η-length sequence so that $p' \upharpoonright \xi = p$ and $p'(i) = \mathbf{1}_{\dot{\mathbb{Q}}_i}$ for all $i \geq \xi$, then $p' \in \mathbb{P}_\eta$. Let $e^\eta_\xi : \mathbb{P}_\xi \to \mathbb{P}_\eta$ denote the embedding $p \mapsto p'$.
(6) $\mathbf{1}_{\mathbb{P}_\xi}$ is the sequence $\left\langle \mathbf{1}_{\dot{\mathbb{Q}}_i} : i < \xi \right\rangle$.
(7) For $p, p' \in \mathbb{P}_\xi$ we have $p \leq p'$ iff $p \upharpoonright i \Vdash_{\mathbb{P}_i} p(i) \leq p'(i)$ for all $i < \xi$.
(8) If $\xi + 1 \leq \alpha$ then $\mathbb{P}_\xi$ is the set of all $p^\frown \dot{q}$ so that $p \in \mathbb{P}_\xi$ and $\dot{q} \in \operatorname{dom}(\dot{\mathbb{Q}}_\xi)$ and $p \Vdash_{\mathbb{P}_\xi} \dot{q} \in \dot{\mathbb{Q}}_\xi$.

The way to think of this: you want to force with $\mathbb{Q}_0$ then $\mathbb{Q}_1$ then $\mathbb{Q}_2$ and so on, for $\mathbb{Q}_i$ with $i < \alpha$. The $\mathbb{P}_\xi$ then represent the iteration of the first ξ many of these, so that $\mathbb{P}_\alpha$ is the total iteration.

Fact 70. *Each e^η_ξ is a complete embedding.*

Proof. Exercise. □

Note that this definition is underdetermined at limit stages. We require, if ξ is limit, that each condition in $\mathbb{P}_\xi$ has its initial segments in $\mathbb{P}_i$ for $i < \xi$. And we require that we can always freely append $\mathbf{1}$s onto a condition in $\mathbb{P}_i$ to get a condition in $\mathbb{P}_\xi$. But we don't say what the support is, that is how often conditions in $\mathbb{P}_\xi$ may have coordinates which are not $\mathbf{1}$s. Similar to product forcing, there will be different choices for support and which one is appropriate will depend on individual circumstances.

Let me state without proof a couple facts about iterated forcing. For proof, consult Kunen or any other standard text.

Fact 71. *Here, "$\dot{\mathbb{Q}}_\xi$ has the ccc" abbreviates "$\mathbf{1}_{\mathbb{P}_\xi} \Vdash \mathbb{Q}_x$ has the ccc", and similarly for other statements.*

- *If each $\dot{\mathbb{Q}}_\xi$ in a finite-support iteration has the ccc, then the entire iteration has the ccc.*
- *If each $\dot{\mathbb{Q}}_\xi$ is a finite-support iteration of length $\geq \omega$ is nontrivial, then the entire iteration adds reals.*

We will not have time in this course to delve into iterated forcing, but I wanted to (briefly!) introduce it to you, as it is greatly important to the theory and application of forcing.

6. Class forcing and changing the continuum function globally

We begin this section with an observation.

Observation 72. *Let $\mathbb{P}$ be a poset. Then $\mathbb{P}$ can only affect the continuum function on an initial segment.*

Proof. Let $\kappa = |\mathbb{P}|$. In particular, $\mathbb{P}$ has the κ^+-cc. So $\mathbb{P}$ won't collapse cardinals $> \kappa$.

Now recall facts about nice names. Namely, if μ is a cardinal then there are at most $(\kappa^\kappa)^\mu$ many nice names for subsets of σ, using that $|\check{\mu}| = \mu$. For large enough μ, we have $(\kappa^\kappa)^\mu = 2^\mu$. (Exercise: check this!) So for large enough μ we get that $\mathcal{P}(\mu)$ in the forcing extension $V[G]$ can have cardinality at most $(2^\mu)^V$. And since no cardinals are collapsed, we have that $(2^\mu)^{V[G]} \geq (2^\mu)^V$. So they must be equal. □

Exercise 73. Get a better lower bound above which $\mathbb{P}$ cannot affect the continuum function than the above-given bound of some large enough cardinal.

So if we want to change the continuum function on a proper class, we cannot force with a set-sized poset. The solution is to use *class forcing*, that is forcing with proper-class sized posets.

There are details to be worked out to make sure things work in this context. Indeed, some facts from forcing with set-sized posets don't carry over to class forcing. For example, there is a class-sized poset which doesn't admit a definable forcing relation for atomic formulae. Forcing with proper classes can destroy the axioms of set theory, as illustrated by the following exercise.[2]

[2]For further details, check out "Characterizations of pretameness and the Ord-cc" by Peter Holy, Regula Krapf, and Philipp Schlicht, arXiv.LO:1710.10825.

Exercise 74. Consider the class partial order $\mathbb{P}$ so that conditions are of the form $p = (d_p, e_p, f_p)$ where d_p is a finite subset of ω, e_p is an extensional digraph on d_p, and $f_p : d_p \to V$ is a function so that $n \mathrel{e_p} m$ implies $f_p(n) \in f_p(m)$. Order $\mathbb{P}$ by extension. That is, $p \leq q$ if $d_p \supseteq d_q$, e_p extends e_q, and f_p extends f_q. Assume $G \subseteq \mathbb{P}$ is V-generic. Show that if $E = \bigcup\{e_p : p \in G\}$ and $F = \bigcup\{f_p : p \in G\}$ then $F : (\omega, E) \to (V, \in)$ is an isomorphism. Conclude that class forcing can be weird.

However, if we restrict to special instances of class forcing, then we have a nice theory. We will focus on Ord-length products of set-sized posets, though we can in fact be a bit more general.[3]

Fact 75. *Let $\mathbb{P}$ be an* Ord*-length product of set-sized posets, where the support of each $p \in \mathbb{P}$ is a set. (So each $p \in \mathbb{P}$ is a set and $\mathbb{P}$ is actually a class, not a "hyperclass" whose elements are themselves classes.) Then $\mathbb{P}$ admits definable forcing relations which satisfy the truth lemma. Moreover, forcing with $\mathbb{P}$ preserves* ZFC.

I won't prove this, due to lack of time. The point is, the restriction on $\mathbb{P}$ allows a definable forcing relation. And then one can in this setting reprove all the basic facts about forcing.

Let us now see how to use class forcing to globally control the behavior of the continuum function. First, let us finally see how to force the GCH.

Theorem 76. *Let $\mathbb{P} = \prod_{\alpha \in \mathrm{Ord}} \mathrm{Col}(\beth_\alpha^+, \beth_{\alpha+1})$ be the Easton support product. Then forcing with $\mathbb{P}$ forces the* GCH.

Proof sketch. Let $G \subseteq \mathbb{P}$ be V-generic. We prove by induction that $V[G] \models \forall \alpha\ \aleph_\alpha = \beth_\alpha$. The base case is true by definition and the limit case is immediate. So all the work is in the successor case. The idea is to split $\mathbb{P}$ into $\mathrm{Col}(\beth_\alpha^+, \beth_{\alpha+1})$ and the rest of the product, call it $\mathbb{P}^{\neg\alpha}$. Then show that $\mathbb{P}^{\neg\alpha}$ doesn't collapse any cardinals in the interval $[\beth_\alpha, \beth_{\alpha+1}]$. And only $\mathrm{Col}(\beth_\alpha^+, \beth_{\alpha+1})$ can affect the value of $\beth_{\alpha+1} = 2^{\beth_\alpha}$, and it forces $2^{\beth_\alpha} = \beth_\alpha^+$. Since by inductive hypothesis we have $\beth_\alpha = \aleph_\alpha$ we then get that $\beth_{\alpha+1} = {\aleph_\alpha}^+ = \aleph_{\alpha+1}$. □

Now that we know how to force GCH, let's see how we can make it fail. This extends our previous work with Easton forcing, where now we want to deal with class-sized products. We will assume GCH in the ground model, for simplification of the presentation. Similar constructions can be done without this assumption, but you have to do more mucking about with cardinal arithmetic. And since we know how to force GCH, this won't hurt the generality of our consistency results.

Theorem 77 (Easton)**.** *Assume* GCH. *Let E be an Easton index function whose domain is the class of regular cardinals. That is, for each regular κ we have $\mathrm{cof}(E(\kappa)) > \kappa$ and $\kappa < \lambda$ implies $E(\kappa) \leq E(\lambda)$. Then if $G \subseteq \mathbb{P}(E)$ is V-generic we have that in $V[G]$ that $2^\kappa = E(\kappa)$ for all regular κ.*

In other words, we can force the continuum function to behave however we like on the regular cardinals.

Proof sketch. This is just like the case with set-sized Easton forcing, except now we are changing things globally. As in that case, you can show that only the multiplicand of $\mathbb{P}(E)$ at κ affects the value of 2^κ. □

[3]The broadest collection of class forcings which admit a nice theory are the *pre-tame* forcings. They are precisely those forcings which preserve ZFC − Powerset and which admit definable forcing relations, among other nice properties. See the above cited paper, or Sy-David Friedman's chapter in *The Handbook of Set Theory*.

As another application of class forcing, let's see that we can code every set into the continuum function.

Definition 78. Let α be an ordinal and λ be an ordinal. Let $g : \mathrm{Ord} \to \mathrm{Ord}^2$ be the inverse of the Gödel pairing function. Define the set $c(\alpha, \lambda) \subseteq \lambda$ as $i \in c(\alpha, \lambda)$ if $2^{\aleph_{\alpha+i+1}} = \aleph_{\alpha+i+2}$ for all $i < \lambda$. Let x be a set. Say that x is *coded into the continuum pattern at α with length λ* if $\in \restriction \mathrm{tc}(\{x\}) \cong g''c(\alpha, \lambda)$.

If x is coded into the continuum pattern at α with length λ for some α and λ then we say x is *coded into the continuum pattern.*

We think of the continuum pattern on the successor cardinals as an Ord-length binary sequence: 1 for yes GCH holds at this place, 0 for no GCH does not hold at this place. Then x is coded into the continuum pattern if some contiguous subsequence of this binary sequence gives an isomorphic copy of the membership relation restricted below x.

Theorem 79. *There is a class forcing $\mathbb{P}$ so that forcing with $\mathbb{P}$ forces every set to be coded into the continuum pattern.*

Proof, with some sketchy bits. We assume GCH. If it doesn't already hold, then force it.

Given posets $\mathbb{A}, \mathbb{B}$, the *lottery sum* of $\mathbb{A}$ and $\mathbb{B}$ is the forcing $\mathbb{A} \oplus \mathbb{B}$ obtained by taking disjoint copies of $\mathbb{A}$ and $\mathbb{B}$ and adding a top element $\mathbf{1}_{\mathbb{A}\oplus\mathbb{B}}$ which is above all conditions in $\mathbb{A}$ and all conditions in $\mathbb{B}$. Observe that a generic for $\mathbb{A} \oplus \mathbb{B}$ picks one of the two posets, as if by lottery, and then gives a generic for that poset. This may look like a trivial or useless construction, but as we will see in the course of proving this theorem, lottery sums can be powerful tools.

We now ready define $\mathbb{P}$, which will be an Ord-length iteration. Namely, $\mathbb{P} = \mathbb{P}_{\mathrm{Ord}}$ from the full-support iteration $\langle \mathbb{P}_\xi : \xi \le \mathrm{Ord} \rangle$, $\left\langle \dot{\mathbb{Q}}_\xi : \xi < \mathrm{Ord} \right\rangle$ defined so that $\dot{\mathbb{Q}}_\xi$ is a $\mathbb{P}_\xi$-name for the lottery sum of trivial forcing and $\mathrm{Add}(\aleph_{\xi+1}, \aleph_{\xi+3})$. In other words, at stage ξ of this iteration we generically choose whether to do nothing, preserving GCH at $\aleph_{\xi+1}$, or to force GCH to fail at $\aleph_{\xi+1}$. By full-support, we mean that at each limit stage η that conditions $p \in \mathbb{P}_\eta$ can be non-$\mathbf{1}$ for arbitrarily many $i < \eta$.

Similar to how Ord-length products of set-sized posets preserve ZFC, so do to Ord-length iterations. (I will not give a proof.) Let $G \subseteq \mathbb{P}$ be V-generic. Then $V[G]$ satisfies ZFC. We want to see that every set in $V[G]$ is coded into the continuum pattern. Note that it suffices to see that for every set of ordinals $a \in V[G]$ with $a \subseteq \gamma$ we have that there is $\alpha \in \mathrm{Ord}$ so that for $i < \gamma$ we have $i \in a$ iff $2^{\aleph_{\alpha+i+1}} = \aleph_{\alpha+i+2}$. Summarize this by saying that a is coded into the continuum pattern.[4]

Claim 80. $\mathbb{P}$ doesn't collapse cardinals.

I will skip a proof. The key point is, by GCH each factor $\dot{\mathbb{Q}}_\xi$ will not not collapse cardinals.

Claim 81. For every $x \in V[G]$ there is $\xi \in \mathrm{Ord}$ so that $x \in V[G_\xi]$ where $G_\xi \subseteq \mathbb{P}_\xi$ is the restriction of G to $\mathbb{P}_\xi$.

I will skip on a proof of this. To briefly sketch the idea: the stages $\dot{\mathbb{Q}}_\xi$ are progressively more and more closed. And since we are using full support, the tail forcings consisting of $\dot{\mathbb{Q}}_\xi$ for a tail of ordinals ξ will preserve that closure. Since κ-closed forcings can't add sets of size $< \kappa$, this means

[4] This contradicts with the previous usage of this phrase, but every set is coded into the continuum pattern, in the previous sense, iff every set of ordinals is coded into the continuum pattern in this new sense, so it's not too terrible an abuse.

that sets in $V[G]$ cannot be added by a sufficiently closed tail, so they must show up in initial segments.

Fix $a \in V[G]$ a set of ordinals with $a \subseteq \gamma$. Fix ξ so that $a \in V[G_\xi]$. I claim that densely many conditions in the tail forcing above ξ will force a to be coded into the continuum pattern. To see this, take a condition p in the the tail forcing. Let $\alpha = \sup\{i : p(i) \neq \mathbf{1}\}$. Then $\alpha \in \mathrm{Ord}$ by set-support. Extend p to the condition p' by not changing anything below α and, for each $i < \gamma$, extending $p(\alpha+i)$ so that $p'(\alpha+i)$ is in the trivial part if $i \notin a$ and $p'(\alpha+i)$ is in the Cohen-forcing part if $i \in a$. Then, any generic containing p' will code a into the continuum pattern starting at α.

So by density, if G^ξ is the restriction of G to the tail forcing, then G^ξ will force a to be coded into the continuum pattern. Thus, we have seen that for an arbitrary set of ordinals $a \in V[G]$ that a is coded into the continuum pattern in $V[G]$. So every set is coded into the continuum pattern, as desired. □

Remark 82. The reason for using an iteration instead of a product is that we want all new sets to be coded into the continuum pattern. If we made an analogous argument with a product instead of an iteration then we'd get that each set in V is coded into the continuum pattern of $V[G]$, but not that every set in $V[G]$ is coded into the continuum pattern. Indeed, by using that the product is *densely weakly homogeneous*[5] we can show that the only sets in $V[G]$ coded into the continuum pattern are those in V.

This idea of coding sets into the continuum pattern can be exploited to prove other results. One such result I leave to you as an exercise.

Exercise 83. Show that the definition $\varphi(x)$ set $x \subseteq \omega$ as $i \in x$ iff $2^{\aleph_{i+1}} = \aleph_{i+2}$ is universal in the following sense: given any real $a \subseteq \omega$ there is a poset $\mathbb{P}_a$ so that forcing with $\mathbb{P}_a$ doesn't add any reals and in the forcing extension by $\mathbb{P}_a$ we have $\varphi(a)$ holds.

That is, there is a single definition for a real so that given any real you can find a forcing extension in which that real is definable by that definition.

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[5]See page 33 of Fuchs, Hamkins, and Reitz, "Set-theoretic geology" for a definition.