MATH655 LECTURE NOTES: PART 2.1 AN INTRODUCTION TO FORCING

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We turn in this part of the course to our next major topic, namely forcing. Introduced by Paul Cohen to prove the consistency of the failure of the continuum hypothesis, forcing has proven to be a remarkably flexible tool, and has become of central importance in set theory.

Some additional references on forcing:

- Timothy Chow, "A beginner's guide to forcing", arXiv[math.LO]:0712.1320.
- Kenneth Kunen, *Set Theory*, chapter IV.
- Akihiro Kanamori, *The Higher Infinite*, section 10.
- J. R. Shoenfield, "Unramified forcing", *Axiomatic Set Theory*, Proc. Sympos. Pure Math., XIII, Part I, Amer. Math. Soc., pp. 537–381.
- Thomas Jech, *Set Theory*, 3rd ed., chapters 14–15.

Chow's paper is, as the name suggests, a first overview of the big ideas of forcing. I strongly advise reading it before jumping into the technical details. My notes here will largely follow Kunen's approach, with the exception of the forcing with sets of ordinals stuff I put in at the beginning. Kanamori does not provide full details, but his approach is oriented toward the interactions between forcing and large cardinals, which is the topic of Part 3 of this course. Shoenfield's paper is a classic, and most modern treatments of forcing owe a debt to it. Finally, Jech's treatment of forcing is based on the boolean algebra approach to forcing, as opposed to the poset-based approach we will take. The two approaches are equivalent, but it is helpful to see both of them, as they have different strengths/weaknesses. The poset approach tends to be more useful in applications of forcing, whereas the boolean algebra approach is usually better for proving theorems about forcing.

1. M-generics and a sketch of forcing

Let M be a countable transitive model of ZFC. Let $\mathbb{P} \in M$ be a poset. Recall the following definition from the poset exercises: Say that a filter $G \subseteq \mathbb{P}$ is *M-generic* if G is $\mathcal{D}_M$-generic where $\mathcal{D}_M$ is the collection of all dense subsets of $\mathcal{D}$ which are in M.

We will see in the sequel that given $G \subseteq \mathbb{P}$ an M-generic there is a smallest model of ZFC extending M which has the same ordinals and contains G. This model, called $M[G]$, is the *forcing extension of M by G*. The poset $\mathbb{P}$ will determine the properties of $M[G]$, and in fact the properties of $M[G]$ will be controlled by certain relations—the *forcing relations*—in the ground model M. On the other hand, the genericity of G will ensure that $M[G]$ is not too dissimilar from M. In particular, it will be used to show that $M[G]$ really does satisfy the axioms of ZFC.

For example, $\mathbb{P}$ might be chosen to make it so that the continuum is larger than $\aleph_1$. In this way, we produce a model $M[G]$ of ZFC where CH is false, showing that ZFC cannot prove the continuum hypothesis. And we could instead choose $\mathbb{P}$ to force the continuum to be $\aleph_1$, there getting a model $M[G]$ of ZFC where CH is true. Altogether, we get that ZFC does not settle the continuum hypothesis one way or the other.

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After we've gone through the work of understanding forcing over countable transitive models we will discuss how to avoid the assumption of the existence of a countable transitive model of ZFC. After all, the existence of such is ε stronger than assuming the consistency of ZFC, and we would like to not have to make extra assumptions to prove these independence results. That is, we would like to be able to conclude that if ZFC is consistent then so are $\mathsf{ZFC} + \mathsf{CH}$ and $\mathsf{ZFC} + \neg\mathsf{CH}$, without needing to assume anything stronger.

2. Names for sets of ordinals

We will not jump straight into the full generality of forcing. Instead, we will first dip our toes into the kiddie pool that is sets of ordinals. It in fact turns out that this "limited" look suffices to see the whole picture, so this kiddie pool is really the whole deep ocean. (We shan't fully pursue this, however.) This idea of doing forcing just by looking just at sets of ordinals is due to Sy Friedman, and it is his ideas I am heavily cribbing from in this section.

Throughout this section fix countable transitive $M \models \mathsf{ZFC}$ and a poset $\mathbb{P} \in M$.

Definition 1 (S. Friedman)**.** A $\mathbb{P}$-*name for a set of ordinals* is a set σ whose elements are all of the form (α, p) with $\alpha \in \mathrm{Ord}$ and $p \in \mathbb{P}$.

Definition 2. Let $x \in M$ be a set of ordinals. Then the *check name* for x is

$$\check{x} = \{(\alpha, \mathbf{1}) : \alpha \in x\}.$$

The idea is that $\mathbb{P}$-name describes a possible set which exists in a forcing extension of M. Of course, the $\mathbb{P}$-name itself is in M, so it alone does not determine this possible set. We need some extra information. This information comes in the form of an M-generic G.

Definition 3. Let σ be a $\mathbb{P}$-name for a set of ordinals and let $G \subseteq \mathbb{P}$ be M-generic. Then the *interpretation of* σ *by* G is $\sigma_G = \{\alpha : \exists p \in G\ (\alpha, p) \in G\}$.

The following exercise illustrates that the choice of generic matters in the interpretation of a $\mathbb{P}$-name.

Exercise 4. Let $\mathbb{P} = {}^{<\omega}2$ be the infinite binary tree. Consider the $\mathbb{P}$-name

$$\sigma = \{(\alpha, p) : \alpha < \omega_1 \text{ and } p \in \mathbb{P} \text{ and } p(0) = 1 \text{ iff } \alpha \text{ is limit}\}.$$

Find M-generics $G_0, G_1 \subseteq \mathbb{P}$ so that $\sigma_{G_0} = \{\alpha < \omega_1 : \alpha \text{ is not limit}\}$ and $\sigma_{G_1} = \{\alpha < \omega_1 : \alpha \text{ is limit}\}$.

For the σ from that exercise, regardless of what the generic was, σ_G would always be in M (though M itself doesn't have enough information to know which of its sets σ will interpret to). But in general this need not happen.

Exercise 5. Again let $\mathbb{P}$ be the infinite binary tree. Consider the $\mathbb{P}$-name

$$\sigma = \{(n, p) : n \in \omega \text{ and } p \in \mathbb{P} \text{ and } p(n) = 1\}.$$

Show that if G is M-generic then $\sigma_G \notin M$.

Given $G \subseteq \mathbb{P}$ an M-generic filter, the sets of ordinals of the forcing extension $M[G]$ are those of the form σ_G for $\sigma \in M$ a $\mathbb{P}$-name for a set of ordinals. And the ordinals of $M[G]$ are the same as the ordinals of M.

The next topic is to determine the connection between $=$ and $\in$ in $M[G]$ and the ground model M. We are focusing here on sets of ordinals. Since the ordinals are fixed between M and $M[G]$, when determining whether $\alpha \in \sigma_G$ there is no complication with α. All the work is on the right-hand side. And notice that once we have figured out how membership works, we get equality for free, using extensionality. So let's think about how to check whether $\alpha \in \sigma_G$, based upon looking at conditions in $\mathbb{P}$.

Start by observing that if $(\alpha, p) \in \sigma$ and $p \in G$ then immediately we get that $\alpha \in \sigma_G$. So we don't need to know everything about G to determine whether $\alpha \in \sigma_G$. We only need to know that p is in. We say that p *forces* $\alpha \in \sigma$ and we write $p \Vdash \alpha \in \sigma$. And this definition is entirely independent of G, so it makes sense to write $p \Vdash \alpha \in \sigma$ without any reference to the generic filter.

But in general, we won't be so lucky to get $(\alpha, p) \in \sigma$, while we still have $\alpha \in \sigma_G$ for any $G \ni p$. To illustrate this, consider the $\mathbb{P}$-name $\sigma = \{(n, p) : n \in \omega \text{ and } \text{ and } p(0) = 1\, |p| \geq 2\}$ where $\mathbb{P} = {}^{<\omega}2$ is the infinite binary tree. Observe that if $\langle 1 \rangle \in G$, then $\sigma_G = \omega$. This is because of $\langle 1 \rangle$ is in G then because it is dense for a condition to have length at least 2, it must be that G contains a condition of length at least 2 which extends $\langle 1 \rangle$. So $\langle 1 \rangle$ forces $n \in \sigma$ for all $n < \omega$, even though $(n, \langle 1 \rangle) \notin G$. To handle this sort of possibility we want to define $p \Vdash \alpha \in \sigma$ to be more general than $(\alpha, p) \in \sigma$.

Definition 6. Let $\mathbb{P}$ be a poset. Then the trinary relation $p \Vdash \alpha \in \sigma$ between conditions $p \in \mathbb{P}$, $\alpha \in \mathrm{Ord}$, and σ a $\mathbb{P}$-name for a set of ordinals is defined as:

$$p \Vdash \alpha \in \sigma \qquad \text{iff} \qquad \text{there are densely many } q \leq p \text{ with } r \geq q \text{ so that } (\alpha, r) \in \sigma.$$

Phrased in pure symbols, this is:

$$p \Vdash \alpha \in \sigma \qquad \text{iff} \qquad \forall q' \leq p\ \exists q \leq q'\ \exists r \geq q\ (\alpha, r) \in \sigma.$$

The intuition is: If p is in a generic G and a set $D \in M$ is dense below p, then G must meet D. This is why we ask for densely many $q \leq p$ to have this property. And the reason we ask for a condition r weaker than q is that generics are filters, hence closed upward. (We could have instead required that $\mathbb{P}$-names have a downward closure property—if $(\alpha, p) \in \sigma$ and $q \leq p$ then $(\alpha, q) \in \sigma$. The definition I gave is the standard way to handle this, and it's a little nicer to work with. In particular, the other way would give problems with handling forcing with proper class posets, which we sometimes have occasion to do.) We will see that this exactly captures what it means for any $G \ni p$ to have $\alpha \in \sigma_G$.

Before we get to that, a few things to note. First off, this definition is Σ_0, hence absolute between transitive sets. Second, note that as there are proper class many ordinals and proper class many $\mathbb{P}$-names, so this is a class relation. So if $M \models \mathsf{ZFC}$ is transitive then $\Vdash^M = \Vdash \cap M$ is a proper class from M's perspective.

Proposition 7. *Suppose $M \models \mathsf{ZFC}$ is transitive and $G \subseteq \mathbb{P} \in M$ is M-generic. Let $\sigma \in M$ be a $\mathbb{P}$-name for a set of ordinals. Then, for all $\alpha \in \mathrm{Ord}^M$ and any M-generic G, we have $\alpha \in \sigma_G$ iff there is $p \in G$ so that $p \Vdash \alpha \in \sigma$.*

Proof. ($\Leftarrow$) Fix $p \in G$ so that $p \Vdash \alpha \in \sigma$. Let D consist of all the conditions incompatible with p and the conditions $q \leq p$ so that there is $r \geq q$ so that $(\alpha, r) \in \sigma$. Then $D \in M$ is dense. Because G is a filter if it meets D it must do so below p. And because G is generic it does meet D. So pick $q \leq p$ in $D \cap G$. Let $r \geq q$ witness that $q \in D$. Then, because G is upward closed we get that $r \in G$. So then, by the definition of σ_G, we have that $\alpha \in \sigma_G$.

($\Rightarrow$) Suppose $\alpha \in \sigma_G$. Then, by the definition of the interpretation of names, there is $(\alpha, p) \in \sigma$ so that $p \in G$. I claim that $p \Vdash \alpha \in \sigma$. To see this, note that if $r = p$ then for any $q \leq p$ we have $r \geq q$ with $(\alpha, r) \in \sigma$. □

We can also define the forcing relation for equality.

Definition 8. Let $\mathbb{P}$ be a poset. Then the trinary relation $p \Vdash \sigma = \tau$ between conditions $p \in \mathbb{P}$ and σ, τ two $\mathbb{P}$-names for sets of ordinals is defined as:

$$p \Vdash \sigma = \tau \qquad \text{iff} \qquad \forall \alpha \in \operatorname{dom}(\sigma) \cup \operatorname{dom}(\tau)\ \forall q \leq p\ (q \Vdash \alpha \in \sigma \Leftrightarrow q \Vdash \alpha \in \tau).$$

Because $\mathbb{P}$-names are relations, it makes sense to talk about their domain. So saying $\alpha \in \operatorname{dom}(\sigma)$ is just a quick way of saying that there is p so that $(\alpha, p) \in \sigma$.

Observe that this definition is Σ_0 and hence absolute between transitive sets.

Proposition 9. *Suppose $M \models \mathsf{ZFC}$ is transitive and $G \subseteq \mathbb{P} \in M$ is M-generic. Let $\sigma, \tau \in M$ be $\mathbb{P}$-names for sets of ordinals. Then, for any M-generic G we have that $\sigma_G = \tau_G$ iff there is $p \in G$ so that $p \Vdash \sigma_G = \tau_G$.*

Proof. ($\Rightarrow$) Suppose $\sigma_G = \tau_G$. Consider the set $D \subseteq \mathbb{P}$ consisting of $p \in \mathbb{P}$ satisfying one of the following three cases:

- $p \Vdash \sigma = \tau$;
- There is $\alpha \in \operatorname{dom}(\sigma) \cup \operatorname{dom}(\tau)$ so that $p \Vdash \alpha \in \sigma$ but every $q \leq p$ has $q \nVdash \alpha \in \tau$; or
- There is $\alpha \in \operatorname{dom}(\sigma) \cup \operatorname{dom}(\tau)$ so that $p \Vdash \alpha \in \tau$ but every $q \leq p$ has $q \nVdash \alpha \in \sigma$.

Clearly $D \in M$, since these are all absolute notions. And D is dense. (Exercise: show this!) So there is $p \in G \cap D$. This p must fall into one of the three cases. In the first case we are done. For the second case, we then get that every $q \in G$ has $q \nVdash \alpha \in \tau$. So, by the earlier proposition about forcing membership, $\alpha \notin \tau_G$. But also by that proposition $\alpha \in \sigma_G$, contradicting that $\sigma_G = \tau_G$. So the second case is impossible. A similar argument shows that the third case is also impossible, leaving the first case as the only possibility.

($\Leftarrow$) Suppose $p \in G$ so that $p \Vdash \sigma_G = \tau_G$. Now fix $\alpha \in \sigma_G$. We know that $\alpha \in \sigma_G$ because there is $p' \in G$ so that $p' \Vdash \alpha \in \sigma$. So there are densely many $q \leq p'$ with $r \geq q$ so that $(\alpha, r) \in \sigma$. By the genericity of G plus the fact that G is upward closed, we get such $r \in G$. Thus, we have $r \in G$ so that $(\alpha, r) \in \sigma$. By the directedness of G take $s \leq r, p$. By the definition of $p \Vdash \sigma = \tau$ plus the fact that $s \Vdash \alpha \in \sigma$ we get that $s \Vdash \alpha \in \tau$. But then $\alpha \in \tau_G$. Altogether, we have seen that $\sigma_G \subseteq \tau_G$. And the other direction holds by the same reasoning, but with σ and τ swapped. $\square$

We want to inductively carry this upward, defining forcing relations $p \Vdash \varphi(\alpha_0, \ldots, \alpha_m, \sigma_0, \ldots, \sigma_n)$ for all formulae $\varphi(x_0, \ldots, x_m, y_0, \ldots, y_n)$ in the language of set theory which only quantify over ordinals or sets of ordinals. The two previous definitions give us the base case of the induction. It remains to see how to carry out the inductive steps.

Definition 10. The forcing relations $p \Vdash \varphi(\alpha_0, \ldots, \alpha_m, \sigma_0, \ldots, \sigma_n)$ between $p \in \mathbb{P}$, ordinals $\alpha_0, \ldots, \alpha_m$, and $\mathbb{P}$-names $\sigma_0, \ldots, \sigma_n$ for sets of ordinals are recursively defined as follows.

(1) $p \Vdash \alpha \in \sigma$ is defined as above.
(2) $p \Vdash \sigma = \tau$ is defined as above.
(3) $p \Vdash \varphi \wedge \psi$ iff $p \Vdash \varphi$ and $p \Vdash \psi$.
(4) $p \Vdash \neg\varphi$ iff there is no $q \leq p$ so that $q \Vdash \varphi$.
(5) Because of how we are treating ordinals and sets of ordinals differently, we need two cases to handle quantification. In the next section, where we take a uniform approach to forcing,

this will not be an issue.

$$p \Vdash \forall \alpha\ \varphi(\alpha, \dots) \quad \text{iff} \quad \forall \alpha \in \mathrm{Ord}\ p \Vdash \varphi(\alpha, \dots)$$
$$p \Vdash \forall x\ \varphi(x, \dots) \quad \text{iff} \quad \forall \sigma \text{ a } \mathbb{P}\text{-name for a set of ordinals } p \Vdash \varphi(\sigma, \dots)$$

Several remarks are in order. We did not give an inductive step for disjunctions or existential quantifiers. But that is okay, since every formula is logically equivalent to one with neither disjunction nor existential quantification. That said, you can directly define how those steps go through.

Exercise 11. Give a recursive definition for $p \Vdash \varphi \vee \psi$ in terms of $p \Vdash \varphi$ and $p \Vdash \psi$ so that $p \Vdash \varphi \vee \psi$ iff $p \Vdash \neg(\neg\varphi \wedge \neg\psi)$. Give a recursive definition for $p \Vdash \exists x\ \varphi(x, \dots)$ in terms of $p \Vdash \varphi(\sigma, \dots)$ so that $p \Vdash \exists x\ \varphi(x, \dots)$ iff $p \Vdash \neg\forall x\ \neg\varphi(x, \dots)$.

Next, I want to highlight that the induction here takes place in the metatheory. Really, what we defined is a family of relations $p \Vdash \varphi(\dots)$ for each formula φ in the language of set theory. But there cannot be a single uniform definition that works for all φ, as we will prove later. Nevertheless, if we fix a single formula φ then the relation $p \Vdash \varphi(\dots)$ is definable. This is because the base case is definable, and there are only finitely many subcases for a fixed φ. So we get a single formula which defines the relation $p \Vdash \varphi(\dots)$, with more complicated φ giving rise to a more complicated definition.

The following result relates the forcing relations to truth in the forcing extension.

Lemma 12 (Truth lemma)**.** *Fix a formula $\varphi(x_0, \dots, x_m, y_0, \dots, y_n)$ in the language of set theory which only quantifies over ordinals or sets of ordinals. Also fix ordinals $\alpha_0, \dots, \alpha_m$ and $\mathbb{P}$-names $\sigma_0, \dots, \sigma_n$ for sets of ordinals. Let G be M-generic. Then*

$$M[G] \models \varphi(\alpha_0, \dots, \alpha_m, (\sigma_0)_G, \dots, (\sigma_n)_G)$$

iff there is $p \in G$ so that

$$p \Vdash \varphi(\alpha_0, \dots, \alpha_m, \sigma_0, \dots \sigma_n).$$

Because we are currently confining ourselves to names for sets of ordinals, we don't know all of $M[G]$. So there's something to be said about making sense of satisfaction for $M[G]$. But since φ only quantifies over ordinals or sets of ordinals and since we know the ordinals and sets of ordinals of $M[G]$, we have enough information to make sense of this.

I won't prove this here, since we will prove the truth lemma in the next section where we look at names for more than just sets of ordinals.

3. Forcing for realsies

We move now to the general setting, where we want to talk about all the sets in the extension, not just the sets of ordinals. The new hurdle is that new sets can have new sets as elements. For example, if $x \in M[G] \setminus M$ then, if $M[G]$ is to satisfy the axiom of Powerset, there must be $\mathcal{P}(x)^{M[G]} \in M[G]$. But clearly this set has x as an element. So we cannot form a name for $\mathcal{P}(x)^{M[G]}$ using just ground model sets as possible elements. Instead, we need to recursively build up names to allow new sets as possible elements. These possible elements are themselves given by names, whence the recursion.

Hereon in this section we work with fixed countable transitive $M \models \mathsf{ZFC}$ and a poset $\mathbb{P} \in M$.

Definition 13. A set is a $\mathbb{P}$-*name* if its elements are all of the form (σ, p) where σ is a $\mathbb{P}$-name and $p \in \mathbb{P}$. Denote by $V^{\mathbb{P}}$ the class of $\mathbb{P}$-names.

It must be made clear why this definition is recursive, and not circular. The reason is that we check whether x is a $\mathbb{P}$-name by looking at the elements of x, which necessarily have lower rank than x does. That is, the well-foundedness of $\in$ is what makes this recursion valid. (So this is another place where Foundation pops up.)

Observe that being a $\mathbb{P}$-name is a Δ_1^{ZFC} property. For the Σ_1-characterization, this is because x is a $\mathbb{P}$-name if there exists a construction tree witnessing the recursive definition carried out on x, and checking that the recursive definition holds at a stage is Σ_0. For the Π_1-characterization, this is because not being a $\mathbb{P}$-name can also be witnessed by a construction tree, but one where we see that a step failed. Accordingly, being a $\mathbb{P}$-name is absolute between transitive models of ZFC. (In fact, much less than ZFC is needed.) Thus, $M^{\mathbb{P}} = (V^{\mathbb{P}})^M = V^{\mathbb{P}} \cap M$.

Let's see some example names:

Definition 14. Let x be a ground model set. Then $\check{x} = \{(\check{y}, \mathbf{1}) : y \in x\}$ is the *check name* for x.

Definition 15. The canonical name for the generic is $\dot{G} = \{(\check{p}, p) : p \in \mathbb{P}\}$.

Exercise 16. Explicitly write down check names for $\emptyset$, for 1, and for 2.

Once we know what $\mathbb{P}$-names are, we need to know how to interpret them.

Definition 17. If G is M-generic and σ is a $\mathbb{P}$-name then the *interpretation of* σ *by* G is $\sigma_G = \{\tau_G : \exists p \in G \ (\tau, p) \in \sigma\}$. Once again, this is a definition by recursion.

Given M and G, $M[G] = \{\sigma_G : \sigma \in M^{\mathbb{P}}\}$ is the *forcing extension of* M *by* G.

Exercise 18. Show that if $\check{x}$ is the check name for x and G is any M-generic then $\check{x}_G = x$. (Hint: prove this by induction on $\in$.)

Exercise 19. Show that $\dot{G}_G = G$ for any M-generic G.

To know all of σ_G, we in general need to know G. But as with names for sets of ordinals, if we merely want to know about one element of σ_G we can do so with only partial information about G. And similarly for checking whether $\sigma_G = \tau_G$.

Definition 20 (Forcing relation for atomic formulae)**.** We define the relations $p \Vdash \sigma \in \tau$, $p \Vdash \sigma \subseteq \tau$, and $p \Vdash \sigma = \tau$ between conditions in $\mathbb{P}$ and $\mathbb{P}$-names by recursion.

$$
\begin{aligned}
p \Vdash \sigma \in \tau \quad &\text{iff} \quad \exists \text{ densely many } q \le p \text{ for which } \exists (\rho, r) \in \tau \text{ so that } q \le r \text{ and } q \Vdash \sigma = \rho \\
p \Vdash \sigma = \tau \quad &\text{iff} \quad \forall \rho \in \operatorname{dom}(\sigma) \cup \operatorname{dom}(\tau)\ \forall q \le p\ (q \Vdash \rho \in \sigma \Leftrightarrow q \Vdash \rho \in \tau).
\end{aligned}
$$

We can unify these two definitions into a single definition, e.g. by tagging which case we are in. The details are routine and uninteresting, so no time will be spent on them. But the point is that we can talk about *the* forcing relation for atomic formulae, rather than having multiple relations.

Like being a $\mathbb{P}$-name, satisfying this recursion is a Δ_1 property. So the forcing relation for atomic formulae is absolute between transitive models of (a fragment of) ZFC.

This exercise establishes a couple immediate facts about the forcing relation for atomic formulae. We will need a bit more machinery to

Exercise 21. Show that for any $p \in \mathbb{P}$ that $p \Vdash \sigma = \sigma$. Show that if $(\tau, r) \in \sigma$ and $p \le r$ then $p \Vdash \tau \in \sigma$.

As before, once we have the forcing relation for atomic formulae we can inductively define forcing relations for every formula. This is exactly the same recursive definition as from the names for sets of ordinals case, except that we only have one type of object to deal with.

Definition 22. The forcing relations $p \Vdash \varphi(\sigma_0, \ldots, \sigma_n)$ between $p \in \mathbb{P}$, and $\mathbb{P}$-names $\sigma_0, \ldots, \sigma_n$ are recursively defined as follows.

(1) $p \Vdash \sigma \in \tau$ is defined as above.
(2) $p \Vdash \sigma = \tau$ is defined as above.
(3) $p \Vdash \varphi \wedge \psi$ iff $p \Vdash \varphi$ and $p \Vdash \psi$.
(4) $p \Vdash \neg\varphi$ iff there is no $q \leq p$ so that $q \Vdash \varphi$.
(5) $p \Vdash \forall x\ \varphi(x, \ldots)$ iff for all $\mathbb{P}$-names σ we have $p \Vdash \varphi(\sigma, \ldots)$.

Once again, the induction takes place in the metatheory. Since we started from the forcing relation for atomic formulae, which we know is definable, we get that for each φ the forcing relation for φ is definable. We will see, however, that they cannot be uniformly definable, for fear of contradicting Tarski.

Before we see that the forcing relation does what we want, we need to check some of its basic properties.

Definition 23. Let $\varphi(x, \ldots)$ be a formula in the language of set theory and $\sigma, \ldots$ be $\mathbb{P}$-names. Then φ *decides* $\varphi(\sigma, \ldots)$, written $p \parallel \varphi(\sigma, \ldots)$ if either $p \Vdash \varphi(\sigma, \ldots)$ or $p \Vdash \neg\varphi(\sigma, \ldots)$.

Definition 24. Let $p \in \mathbb{P}$. Then $D \subseteq \mathbb{P}$ is *dense below* p if given any $q \leq p$ there is $r \leq q$ so that $r \in D$.

Exercise 25. Show that if D is dense below p and $q \leq p$ then D is dense below q. Show that if the collection of $q \leq p$ so that D is dense below q is dense below p, then D is dense below p.

Proposition 26. *Consider a poset $\mathbb{P}$. Then all of the following.*

(1) *Fix $\varphi(x, \ldots)$ and $\mathbb{P}$-names $\sigma, \ldots$. Then $\{p \in \mathbb{P} : p \parallel \varphi(\sigma, \ldots)\}$ is dense.*
(2) *Suppose $p \Vdash \varphi(\sigma, \ldots)$ and $q \leq p$. Then $q \Vdash \varphi(\sigma, \ldots)$.*
(3) *No condition p has $p \Vdash \varphi(\sigma, \ldots)$ and $p \Vdash \neg\varphi(\sigma, \ldots)$.*
(4) *If the conditions $q \leq p$ so that $q \Vdash \varphi(\sigma, \ldots)$ are dense below p, then $p \Vdash \varphi(\sigma, \ldots)$.*

Proof. (1) Suppose it were not the case that the conditions deciding $\varphi(\sigma, \ldots)$ were dense. Then there would be p so that no $q \leq p$ has $q \parallel \varphi(\sigma, \ldots)$. But if there is no $q \leq p$ so that $q \Vdash \varphi(\sigma, \ldots)$, then by definition $p \Vdash \neg\varphi(\sigma, \ldots)$. Contradiction.

(2) First let's prove this for the atomic formulae. Suppose $p \Vdash \sigma \in \tau$ and let's show that $q \leq p$ also forces $\sigma \in \tau$. That is, we want to show there are densely many $r \leq q$ for which there is $(\rho, s) \in \tau$ so that $s \geq r$ and $r \Vdash \sigma = \rho$. There are densely many such $r \leq p$, so this is immediate by the exercise. The argument for $\sigma \subseteq \tau$ is similar, since that is also defined by the existence of densely many conditions below blah blah so that blah blah. And from that we immediately get it for $\sigma = \tau$. It remains to carry it upward to all formulae, which is done by induction. (Exercise: explicitly do this!)

(3) This follows immediately from (2).

(4) This is proved by induction. First for atomic formulae: suppose it is dense below p to force $\sigma \in \tau$. We want to see there are densely many $q \leq p$ with $(\rho, r) \in \tau$ so that $r \geq q$ and $q \Vdash \sigma = \rho$. This follows from the exercise: it is dense below p to have it be dense below you to have a q with (ρ, r) such that blah blah, so this property must be dense below p. The argument for $\sigma \subseteq \tau$ is

similar, and this immediately yields the case for $\sigma = \tau$. Finally, you can inductively prove this for all formulae. (Exercise: do this!) □

Another useful fact is that forcing respects logical equivalence. On the course website you will find an exercise set which will guide you through proving that.

Definition 27. φ and ψ are logically equivalent if $\emptyset \models \varphi \Leftrightarrow \psi$, where $\emptyset$ is the empty theory with no axioms. By the completeness plus soundness theorems, this is equivalent to asking that $\emptyset \vdash \varphi \Leftrightarrow \psi$.

All these preliminaries established, we are now ready to finally see the connection between the forcing relation in M and truth in $M[G]$. Let us begin with atomic formulae.

Proposition 28. *Let G be M-generic. Then $\sigma_G \in \tau_G$ iff there is $p \in G$ so that $p \Vdash \sigma \in \tau$ and $\sigma_G = \tau_G$ iff there is $p \in G$ so that $p \Vdash \sigma = \tau$.*

Proof. We prove this by induction on names in M, working from the perspective of V. This is a valid use of transfinite induction because if $(p, \tau) \in \sigma$ then $\operatorname{rank}(\tau) < \operatorname{rank}(\sigma)$, and thus this induction is over a set-like, well-founded relation.

There are two cases in the induction, whether we want to conclude the equivalence for $\in$ or for $\subseteq$. And we will get the $=$ case as a corollary of the $\subseteq$ case.

($\in, \Rightarrow$) Suppose $\sigma_G \in \tau_G$. This happens iff there is $p \in G$ so that there is $(\rho, p) \in \tau$ with $\rho_G = \sigma_G$. By inductive hypothesis, $\rho_G = \sigma_G$ iff there is $p' \in G$ so that $p' \Vdash \rho = \sigma$. Take $q \in G$ with $q \leq p, p'$. Then, any $r \leq q$ has a condition above, namely p, so that $(\rho, p) \in \tau$ with $r \Vdash \sigma = \rho$. Note that this used that if $p' \Vdash \sigma = \rho$ then so does any condition stronger condition.

($\in, \Leftarrow$) Suppose there is $p \in G$ so that $p \Vdash \sigma_G \in \tau_G$. Then, by density and by the definition of the forcing relation relation for membership, there is $q \in G$ with $q \leq p$ with $(\rho, r) \in \tau$ so that $r \geq q$ and $q \Vdash \sigma = \rho$. By inductive hypothesis, $\sigma_G = \rho_G$. And by the upward closure of G we get that $\rho_G \in \tau_G$. So $\sigma_G = \rho_G \in \tau_G$, as desired.

($=, \Rightarrow$) Suppose $\sigma_G = \tau_G$. Now consider $D \subseteq \mathbb{P}$ defined as the set of all $p \in \mathbb{P}$ so that one of the following:

- $p \Vdash \sigma = \tau$;
- For some $\rho \in \operatorname{dom}(\sigma) \cup \operatorname{dom}(\tau)$ we have $p \Vdash \rho \in \sigma$ but $p \Vdash \rho \notin \tau$; or
- For some $\rho \in \operatorname{dom}(\sigma) \cup \operatorname{dom}(\tau)$ we have $p \Vdash \rho \in \tau$ but $p \Vdash \rho \notin \sigma$.

Clearly $D \in M$, because these notions are all absolute. And D is dense by the definition of the forcing relation for equality and negation. Thus, there is $p \in G \cap D$ by density. Now consider each case. In the first case we are done. In the second case, by inductive hypothesis we get that $\rho_G \in \sigma_G$ but $\rho_G \notin \tau_G$, contradicting that $\sigma_G = \tau_G$. And similarly in the third case we get that $\rho_G \in \tau_G$ but $\rho_G \notin \sigma_G$, also a contradiction. So the first case was the only possibility all along.

($=, \Leftarrow$) Suppose there is $p \in G$ so that $p \Vdash \sigma = \tau$. By definition, this means that for all $\rho \in \operatorname{dom}(\sigma) \cup \operatorname{dom}(\tau)$ and for all $q \leq p$ we have $q \Vdash \rho \in \sigma$ iff $q \Vdash \rho \in \tau$. In particular, there is always such $q \in G$. So by inductive hypothesis $\rho_G \in \sigma_G$ iff $\rho_G \Leftrightarrow \tau_G$. But this is just saying that $\sigma_G = \tau_G$, as desired. □

Now let's prove this for all formulae.

Theorem 29 (Truth lemma)**.** *Fix a formula $\varphi(x_0, \ldots, x_n)$ in the language of set theory and fix $\mathbb{P}$-names $\sigma_0, \ldots, \sigma_n$. Let G be M-generic. Then $M[G] \models \varphi((\sigma_0)_G, \ldots, (\sigma_n)_G)$ iff there is $p \in G$ so that, in M, $p \Vdash \varphi(\sigma_0, \ldots, \sigma_n)$.*

Note that we have to specify that $p \Vdash \varphi(\sigma_0, \ldots, \sigma_n)$ is being evaluated in M because this is not absolute for arbitrary φ, as is easily checked.

Proof. This is proved by induction outside of M, i.e. in V. The base case of the induction, for atomic formulae, was handled by the earlier proposition.

(Conjunction) Exercise. (Hint: remember that "and" means and $\ddot{\smile}$)

(Negation, $\Rightarrow$) Suppose $M[G] \models \neg\varphi$. Then $M \not\models \varphi$. By inductive hypothesis there is no $p \in G$ so that $p \Vdash \varphi$. By the earlier lemma, there is $p \in G$ which decides φ. The only option is that $p \Vdash \neg\varphi$.

(Negation, $\Leftarrow$) Suppose that there is $p \in G$ so that $p \Vdash \neg\varphi$. Suppose toward a contradiction that $M[G] \models \varphi$. Then there would, by inductive hypothesis, be $q \in G$ so that $q \Vdash \varphi$. By the directedness of G there would then be $r \leq p, q$ in G. Because $r \leq q$, we have $r \Vdash \varphi$. But no $r \leq p$ can force φ. Contradiction. So $M[G] \not\models \varphi$ so $M[G] \models \neg\varphi$.

(Quantification, $\Rightarrow$) Suppose $M[G] \models \forall x\ \varphi(x)$. Then, for $\mathbb{P}$-name σ we have $M[G] \models \varphi(\sigma_G)$. By inductive hypothesis, for each $\mathbb{P}$-name σ we then get $p_\sigma \in G$ so that $p_\sigma \Vdash \varphi(\sigma)$. Suppose toward a contradiction that no $p \in G$ has $p \Vdash \forall x\ \varphi(x)$. Then, because it is dense to decide a statement, there must be $p \in G$ so that $p \Vdash \neg\forall x\ \varphi(x)$. Because forcing respects logical equivalence, $p \Vdash \exists x\ \neg\varphi(x)$. Now take $q \in G$ with $q \leq p, p_\sigma$ for some $\mathbb{P}$-name σ. Then, because $q \leq p_\sigma$ we must have that $q \Vdash \varphi(\sigma)$ and hence $q \Vdash \exists x\ \varphi(x)$, since $\varphi(\sigma) \Rightarrow \exists x\ \varphi(x)$ is a logical validity. But also $q \leq p$ so $q \Vdash \neg\exists x\ \varphi(x)$, a contradiction.

(Quantification, $\Leftarrow$) Suppose there is $p \in G$ with $p \Vdash \forall x\ \varphi(x)$. Then, by the definition of the forcing relation, $p \Vdash \varphi(\sigma)$ for each $\mathbb{P}$-name σ. So by inductive hypothesis $M[G] \models \varphi(\sigma_G)$ for each $\mathbb{P}$-name σ. So $M[G] \models \forall x\ \varphi(x)$, as desired.

This is the final step in the argument, completing the proof. □

The fact that there are definable relations $p \Vdash \varphi(\sigma_0, \ldots)$ over M so that truth in $M[G]$ is captured by these relations is known as the *Forcing Theorem*. The way we approached the issue, it was clear that the relations $\Vdash$ were definable, since we explicitly defined them by recursion. An alternative approach, taken by e.g. Kunen, is to first define $(p \Vdash \varphi(\sigma_0, \ldots))^M$ as happening iff $M[G] \models \varphi((\sigma_0)_G, \ldots)$ for all M-generic filters G with $p \in G$. Of course, this definition is a definition in V, not over M, since M cannot see M-generic filters. So if you take this approach you must later give an actual definition of $\Vdash$. I opted to directly give the definition, rather than first giving a pseudodefinition.

Exercise 30. Show that $M \models [p \Vdash \varphi(\sigma_0, \ldots)]$, according to our definition, iff $M[G] \models \varphi((\sigma_0)_G, \ldots)$ for all M-generic filters G with $p \in G$.

The last step in our introduction to forcing is to see that it preserves the basic axioms of set theory. After we are secure in that knowledge we can move on to applications of the technique. I remind you of our assumption that M is a countable transitive model of ZFC. Let us begin with some basic axioms.

Proposition 31. *$M[G]$ always satisfies Extensionality, Infinity, and Foundation.*

Proof. First let us see that $M[G]$ is transitive. So take $x \in \sigma_G \in M[G]$. By the definition of the evaluation of names, we get that $x = \tau_G$ for some $\tau \in \operatorname{dom}(\sigma)$. But then $x \in M[G]$, as desired.

Now that we know $M[G]$ is transitive we immediately get that it satisfies Extensionality and Foundation. It satisfies Infinity because $\check{\omega}_G = \omega \in M[G]$. □

Definition 32. Let σ, τ be $\mathbb{P}$-names. Define the name $\operatorname{up}(\sigma, \tau) = \{(\sigma, \mathbf{1}), (\tau, \mathbf{1})\}$. And define the name $\operatorname{op}(\sigma, \tau) = \operatorname{up}(\operatorname{up}(\sigma, \sigma), \operatorname{up}(\sigma, \tau))$.

Exercise 33. Show that $M[G]$ always satisfies Pairing by showing that $\text{up}(\sigma,\tau)_G = \{\sigma_G, \tau_G\}$. Also show that $\text{op}(\sigma,\tau)_G = (\sigma_G, \tau_G)$.

Proposition 34. *$M[G]$ always satisfies Separation.*

Proof. Fix $\sigma_G \in M[G]$ and let $\varphi(x, \tau_G)$ be a formula in the language of set theory, possibly with a single parameter τ_G. (Since we showed $M[G]$ satisfies Pairing we know that $M[G]$ can code finite tuples and so we can replace multiple arguments with a single tuple of arguments. So one parameter is enough.) We want to see that $a = \{x \in \sigma_G : M[G] \models \varphi(x, \tau_G)\}$ is in $M[G]$. So we need to find $\tau \in M^{\mathbb{P}}$ so that $\tau_G = a$.

Here's what to do. Let τ consist of pairs (ρ, p) so that $\operatorname{rank}\rho < \operatorname{rank}\sigma$ and $p \Vdash \varphi(\rho,\tau) \wedge \rho \in \sigma$. Note that if $\rho'_G \in \sigma_G$ then, by modifying ρ' by throwing elements of high rank, there is ρ with $\rho_G = \rho'_G$ and $\operatorname{rank}\rho < \operatorname{rank}\sigma$. So if we want to form subsets of σ_G it is enough to only consider ρ of lower rank. By the forcing theorem we get for ρ with $\operatorname{rank}\rho < \operatorname{rank}\sigma$:

$$\rho_G \in \tau_G \qquad \text{iff} \qquad \rho_G \in \sigma_G \text{ and } \varphi(\rho_G, \tau_G).$$

So $\tau_G =$ as desired. □

I want to remark on why we did this hootenanny with ρ of low rank. The issue is that for a nontrivial $\mathbb{P}$, given $\sigma \in M^{\mathbb{P}}$ with $\sigma_G \neq \emptyset$ and M-generic G there will always be τ of arbitrarily rank so that $\tau_G \in \sigma_G$. To see this, take any τ' with $\tau_G \in \sigma_G$ and let $q \in \mathbb{P}$ be a condition not in G. Then $\tau = \tau' \cup \{(\check{\alpha}, q)\}$ has rank $> \alpha$ but $\tau_G = \tau'_G \in \sigma_G$. So to have a set-sized $\mathbb{P}$-name, we need to restrict the ranks allowed.

Exercise 35. Show that $M[G]$ always satisfies Union by showing that if σ is a $\mathbb{P}$-name in M then if $\tau = \bigcup \operatorname{dom}\sigma$ we have $\bigcup \sigma_G \subseteq \tau_G$. Use Separation in $M[G]$ to conclude $\bigcup \sigma_G \in M[G]$.

Using Greek letters for names is starting to get tedious. Now's a good time to start using the standard convention in set theory. Common usage when talking about an object $a \in M[G]$ is to write $\dot{a}$ for a name for a. That is, $\dot{a}_G = a$. We already saw an example of this with the canonical name $\dot{G}$ for the generic. The reason for this convention is that when forcing we only care about names insofar as they tell us about the generic. An argument might start something like "let $\dot{f}$ be a name so that $\mathbf{1} \Vdash \dot{f} : \kappa \to \omega_1$". Then $f = \dot{f}_G$ will be a function, and we just think in that way, rather than continually referring to some abstract σ or τ. We will still sometimes use Greek letters to talk about names, when we don't have an interpretation in mind.

Proposition 36. *$M[G]$ always satisfies Replacement.*

Proof. We prove that $M[G]$ satisfies Collection, which is an equivalent formulation of Replacement.[1]

Fix $a \in M[G]$ and assume that in $M[G]$ for every $x \in a$ there is y so that $M[G] \models \varphi(x,y)$, where I suppress writing any parameters being used. Let $\dot{a}$ be a name for a. Working in M, let Q be a set of names so that for all $p \in \mathbb{P}$ and all $\sigma \in \operatorname{dom}\dot{a}$ we have that if there is a name τ so that $p \Vdash \varphi(\sigma,\tau)$ then there is such $\tau \in Q$. Such Q exists by Reflection applied in M. Set $\dot{b} = \{(\tau, \mathbf{1}) : \tau\ inQ\}$.

Let us now see that in $M[G]$ for any $x \in a$ there is $y \in b = \dot{b}_G$ so that $M[G] \models \varphi(x,y)$. To this end, fix $x \in a$. Thus, $x = \sigma_G$ for some $\sigma \in \operatorname{dom}\dot{a}$. We have, by an earlier assumption, that $M[G] \models \varphi(\sigma_G, \tau_G)$ for some G. So by the forcing theorem, in M there is $p \in G$ so that $p \Vdash \varphi(\sigma,\tau)$. Thus there is such $\tau \in Q$. But then $\tau_G \in b$, so we have a witness in b, as desired. □

[1] Actually, Collection is stronger than Replacement in the absence of Powerset. But we will later see that Powerset holds, and they are equivalent in that context.

Proposition 37. *$M[G]$ always satisfies Powerset.*

Proof. We know that $M[G]$ satisfies Separation, so it is enough to check that if $a \in M[G]$ then there is $b \in M[G]$ so that any $x \subseteq a$ from $M[G]$ is in b. To this end, fix a name $\dot{a}$ for a. Define $\dot{b} = \{(\sigma, \mathbf{1}) : \mathrm{dom}(\sigma) \subseteq \mathrm{dom}(\dot{a})\}$. Now let $b = \dot{b}_G$. Let's see that b is as desired. Take $x \subseteq a$ in $M[G]$. Let $\dot{x}$ be a name for x. Now consider the $\mathbb{P}$-name $\tau = \{(\sigma, p) : \sigma \in \mathrm{dom}(\dot{a}) \text{ and } p \Vdash \sigma \in \dot{x}\}$. Note that $\dot{(\tau)} \subseteq \mathrm{dom}(\dot{a})$. So if we can see that $\tau_G = x$ then we are done.

($x \subseteq \tau_G$) Since $x \subseteq a$, every element of x is of the form σ_G for some $\sigma \in \mathrm{dom}(\dot{a})$. But $\sigma_G \in x$ iff there is $p \in G$ so that $p \Vdash \sigma \in \dot{x}$. Then $(\sigma, p) \in \tau$, so $\sigma_G \in \tau_G$.

($\tau_G \subseteq x$) By definition plus the forcing theorem, $\sigma_G \in \tau_G$ iff $\sigma \in \mathrm{dom}(\dot{a})$ and there is $p \in G$ so that $p \Vdash \sigma \in \dot{x}$. But then $\sigma_G \in x$. □

Proposition 38. *$M[G]$ always satisfies Choice.*

Proof. Fix $a \in M[G]$. We want to see that $M[G]$ has a well-order of a. Let $\dot{a} \in M$ be a name for a. Using AC in M, let $\langle \sigma_\xi : \xi < \alpha \rangle$ be a sequence enumerating the elements of $\mathrm{dom}(\dot{a})$. Consider the name

$$\dot{f} = \{(\mathrm{op}(\check{\xi}, \sigma_\xi), \mathbf{1}) : \xi < \alpha\}.$$

Let $f = \dot{f}_G \in M$. Then $f = \langle (\sigma_\xi)_G : \xi < \alpha \rangle$. That is, f is a function with domain α and $a \subseteq \mathrm{ran}\, f$. Now use Separation in $M[G]$ to define a well-order of a as $x < y$ if $\min\{\xi < \alpha : f(\xi) = x\} < \min\{\xi < \alpha : f(\xi) = y\}$. □

Altogether, we have seen:

Theorem 39. *If M is a transitive model of* ZFC *and $G \subseteq \mathbb{P} \in M$ is M-generic then $M[G] \models$* ZFC. □

Next we want to see that $M[G]$ is the smallest transitive model of ZFC which contains M as a subset and G and as element. First, recall that $M \subseteq M[G]$ because $\check{x}_G = x$ for all $x \in M$. And $G \in M[G]$ because $\dot{G}_G = G$.

Proposition 40. $\mathrm{Ord}^M = \mathrm{Ord}^{M[G]}$. *That is, forcing doesn't add new ordinals.*

Proof. First, prove by induction on names to prove that $\mathrm{rank}(\sigma_G) \leq \mathrm{rank}(\sigma)$ for all $\mathbb{P}$-names σ. Now take $\alpha \in \mathrm{Ord}^{M[G]}$. Let $\dot{\alpha} \in M^{\mathbb{P}}$ be a name for α. Then $\alpha = \mathrm{rank}\,\alpha \leq \mathrm{rank}(\dot{\alpha}) \in \mathrm{Ord}^M$. Since Ord^M is transitive this means $\alpha \in \mathrm{Ord}^M$. So we have seen $\mathrm{Ord}^{M[G]} \subseteq \mathrm{Ord}^M$. The other direction follows from the fact that $M \subseteq M[G]$ and that being an ordinal is absolute to transitive models. □

Proposition 41. *Suppose $N \models$* ZFC *has $M \subseteq N$ and $G \in N$ where G is M-generic. Then $M[G] \subseteq N$.*

Proof. Because $M \subseteq N$ we have that $\sigma \in N$ for all $\mathbb{P}$-names $\sigma \in M$. And since $G \in N$ and N satisfies ZFC we get that N can carry out the construction of σ_G for $\sigma \in M$. So $\sigma_G \in N$ for all $\sigma \in M^{\mathbb{P}}$. So $M[G] \subseteq M$. □

Corollary 42. $|M[G]| = |M|$. *In particular, if M is countable so is $M[G]$.*

Proof. By Löwenheim–Skolem if there is $N \supseteq M$ with $G \in N$ then there is such N of cardinality $|M|$. But $M[G]$ must be a subset of it, so it cannot have larger cardinality. □

4. Metamathematical concerns

To cap off this discussion of the mechanics of forcing, let's discuss some metamathematical issues. Let me begin by stating a result we will prove in the near future.

Theorem 43 (Cohen)**.** *There is a definition for a poset $\mathbb{P} \in M$ so that if $M \models \mathsf{ZFC}$ is countable and transitive and if $G \subseteq \mathbb{P}^M$ is M-generic then $M[G] \models 2^{\aleph_0} = \aleph_2$.*

We would like to use this result to show that if ZFC is consistent then so is $\mathsf{ZFC} + \neg\mathsf{CH}$. But as stated, we cannot do that. The issue is that "there is a transitive model of ZFC" is a little bit stronger than "ZFC is consistent". We need a way around this. I will sketch two approaches.

The first goes through a fact we proved back in part 0. Namely, as a corollary of the Levý–Montague reflection principle we saw that ZFC proves the consistency of every finite subtheory of ZFC. Now suppose toward a contradiction that ZFC is consistent but $\mathsf{ZFC} + \neg\mathsf{CH}$ is not consistent. Then, it must be that there is a proof from the axioms of ZFC that CH holds. Proofs are finite objects, so this proof must use only finitely many axioms. Moreover, note that the argument that $M[G] \models 2^{\aleph_0} = \aleph_2$ from Cohen's theorem will only use finitely many axioms of ZFC in M, since this fact will come down to saying that a certain thing is forced, which is a single statement and thus will only need finitely many axioms of ZFC. So let T be a finite fragment of ZFC large enough to capture both these pieces. Then we get $M[G] \models T + \neg\mathsf{CH}$ but also $T \vdash \mathsf{CH}$. This is a contradiction, so our original assumption that $\mathsf{ZFC} + \neg\mathsf{CH}$ were inconsistent must be false. So we were able to get that $\mathsf{ZFC} + \neg\mathsf{CH}$ is consistent just from the assumption that ZFC is consistent.

The second approach is to develop forcing in such a manner that it works over non-transitive models of set theory. The approach we used above does not work in that case. The problem is that interpretation of names σ_G is defined, externally to M, by transfinite recursion. If $\in^M$ is not well-founded, then this recursion is not valid. So this approach does not work for arbitrary models of set theory.

Here's a sketch of the alternative: Let $(M, \in^M)$ be a model of set theory, transitive or otherwise, and suppose $G \subseteq \mathbb{P} \in M$ is M-generic. Now define two relations $=_G$ and $\in_G$ on the $\mathbb{P}$-names in M:

$$\begin{aligned} \sigma =_G \tau \quad &\text{iff} \quad \exists p \in G\ M \models (p \Vdash \sigma = \tau) \\ \sigma \in_G \tau \quad &\text{iff} \quad \exists p \in G\ M \models (p \Vdash \sigma \in \tau) \end{aligned}$$

Note that it makes sense to ask whether $M \models p \Vdash \sigma \in \tau$, even if M isn't transitive, because this is just asking whether M, a model of set theory, satisifies a certain assertion in the language of set theory. One can show that $=_G$ is an equivalence relation on $M^{\mathbb{P}}$ and that $\in_G$ is a congruence modulo $=_G$. One can then define $M[G]$ to have as domain the $=_G$-equivalence classes with $\in_G$ as its membership relation, and show that M canonically embeds into $M[G]$ via the map $x \mapsto \check{x}/{=_G}$.

The following exercise has you show that this approach is equivalent in the case that M is transitive.

Exercise 44. Suppose $M \models \mathsf{ZFC}$ is transitive and that $G \subseteq \mathbb{P} \in M$ is M-generic. Let σ, τ be $\mathbb{P}$-names in M. Show that $\sigma_G = \tau_G$ iff $\sigma =_G \tau$ and show that $\sigma_G \in \tau_G$ iff $\sigma \in_G \tau$.

Equipped with this approach to forcing, we can go from $\mathrm{Con}(\mathsf{ZFC})$ to $\mathrm{Con}(\mathsf{ZFC} + \neg\mathsf{CH})$. Namely, assume $\mathrm{Con}(\mathsf{ZFC})$. Then by Gödel's completeness theorem plus downward Löwenheim–Skolem, there is a countable model M of ZFC. So we can find M-generics, because M is countable, and then use this alternative approach to forcing to carry out Cohen's argument to get $M[G] \models \mathsf{ZFC} + \neg\mathsf{CH}$.

5. Exercises

This collection of exercises settles a dangling thread from earlier. Our discussion of forcing started by looking at Sy Friedman's forcing with names for sets of ordinals, before we moved to the standard approach. I claimed at the time that Friedman's approach is completely general. These exercises return to that point, guiding you through the argument for this.

The first exercise has you prove an important fact about transitive models of set theory, namely that they are determined by their sets of ordinals.

Exercise 45. Suppose M and N are two transitive models of ZFC, either sets or proper classes. Show that $M = N$ iff $\{x \subseteq \mathrm{Ord} : x \in M\} = \{x \subseteq \mathrm{Ord} : x \in N\}$.

Note that if M and N have the same sets of ordinals then $\mathrm{Ord} \cap M = \mathrm{Ord} \cap N$, as ordinals are themselves sets of ordinals.

Next show that whether you do ordinary forcing or forcing with names for sets of ordinals, you get the same sets of ordinals.

Exercise 46. Let $M \models \mathsf{ZFC}$ be transitive and suppose $G \subseteq \mathbb{P} \in M$ is M-generic. Let $M[G]$ denote the ordinary forcing extension of M and let $\mathcal{SO}(M, G)$ be the collection of σ_G, where σ is a name for a set of ordinals. (See the definition from section 2.) Show that $\mathcal{SO}(M, G) = \{x \subseteq \mathrm{Ord} : x \in M[G]\}$.

Let $g : \mathrm{Ord} \to \mathrm{Ord}^2$ denote the inverse of the Gödel pairing function. (Recall that g^{-1} was a bijection which maps pairs of ordinals to ordinals. And it is Δ_1-definable so it is absolute to transitive models of set theory.) So if x is a set of ordinals then $g''x$ is a set of pairs of ordinals, which can be thought of as a relation on a set of ordinals. In this way, working just with sets of ordinals we can talk about relations, functions, and so on. And since we can identify sets x by the isomorphism class of the graph of the membership relation below x, we can recover x just by having enough sets of ordinals.

Exercise 47. Let M and G be as above. Define N to consist of the sets x so that there is $e \in \mathcal{SO}(M, G)$ so that $g''e$ is a well-founded extensional relation and $(\mathrm{tc}(\{x\}), \in)$ is isomorphic to $(\mathrm{dom}(g''e), g''e)$. Show that $N = M[G]$.

As a more open-ended exercise, you can try to come up with axioms to capture what ZFC proves about sets of ordinals.

Exercise 48. Come up with a set of axioms S to describe the ordinals and sets of ordinals of models of set theory. That is, you want that ZFC proves every axiom of S is true of the ordinals/sets of ordinals. Can you get S which is bi-interpretable with ZFC. (Hint: read "The theory of sets of ordinals" by Koepke and Koerwien, where they do just this.)

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