MATH655 LECTURE NOTES: PART 1.2 SUPERCOMPACT CARDINALS AND BEYOND

KAMERYN J. WILLIAMS

The important characterization of measurable cardinals was the following: κ is measurable iff κ is the critical point of an elementary embedding $j : V \prec M$. We saw that M and V agree up to κ. Specifically, ${V_{\kappa+1}}^M = V_{\kappa+1}$ and ${}^\kappa M \subseteq M$. However, this agreement failed drastically once we stepped beyond κ. For instance, $M \models j(\kappa)$ is inaccessible, but in V we can see that $j(\kappa) < (2^\kappa)^+$, and so is far from being inaccessible.

This theme suggests potentially new large cardinal notions. Can we ask for M to more closely resemble V, thereby getting stronger notions? The answer is yes, and large cardinals given by embedding characterizations have been fruitfully studied. Observe that if a large cardinal property is defined by "κ is foo if κ is the critical point of an embedding $j : V \prec M$ so that M satisfies bar and baz", then κ is automatically measurable. So these large cardinal notions all exceed a measurable in strength.

1. Supercompactness

In this section we will focus on supercompact cardinals, possibly the most important of these large cardinals beyond measurables. Let me begin by defining supercompact cardinals in terms of embeddings. Of course, this definition cannot be directly formalized within ZFC, since it is a definition by quantification over a proper class. But we will later extract an equivalent formulation via the existence of certain combinatorial objects, akin to the definition of measurable cardinals by normal measures, giving us a satisfactory formalization within ZFC.

Definition 1 (Solovay–Reinhardt)**.** Let λ be a cardinal. Say that κ is λ-*supercompact* if there is a nontrivial elementary embedding $j : V \prec M$ so that

(1) $\operatorname{crit} j = \kappa$;
(2) ${}^\lambda M \subseteq M$—that is, M is closed under λ-sequences; and
(3) $j(\kappa) > \lambda$.

If κ is λ-supercompact for all λ we say that κ is *supercompact*.

Let's make some easy observations. First, note that if $\lambda' < \lambda$ then κ being λ-supercompact implies κ is λ'-supercompact. Second, note that κ being measurable is equivalent to κ being κ-supercompact. Indeed, we can extract more from the assumption that κ has a bit more supercompactness. As a warm-up, let's see that supercompact cardinals have Mitchell order > 1.

Proposition 2. *Suppose κ is 2^κ-supercompact. Then $o(\kappa) > 1$.*

Proof. Let $j : V \prec M$ witness that κ is 2^κ-supercompact. Define a normal measure $U \subseteq \mathcal{P}(\kappa)$ as $X \in U$ iff $\kappa \in j(X)$. But because M is closed under 2^κ-sequences, every ultrafilter on κ is in M. So $M \models \kappa$ is measurable and hence $\{\alpha < \kappa : \alpha \text{ is measurable}\} \in U$, by the definition of U. So

Date: February 25, 2019.

there is a normal measure on κ which concentrates on the measurable cardinals, witnessing that $o(\kappa) > 1$. □

And now let's push it as far as we can.

Proposition 3. *Suppose κ is 2^κ-supercompact. Then $o(\kappa) = (2^\kappa)^+$.*

Proof. We proceed by induction. Suppose for $\alpha < (2^\kappa)^+$ we have an ultrafilter U_α on κ so that U_α which concentrates on the cardinals $< \kappa$ with Mitchell order β for each $\beta < \alpha$. But then $U_\alpha \in M$, so $M \models o(\kappa) \geq \alpha$. Thus, using the same U as in the previous proposition we get that U concentrates on the cardinals $< \kappa$ with Mitchell order α. And this induction goes through for all $\alpha < (2^\kappa)^+$, so U witnesses that $o(\kappa)$ is as large as can be, namely $(2^\kappa)^+$. □

We saw with measurable cardinals that they exhibit a degree of reflection, where certain properties of κ reflected down to smaller cardinals. Supercompact cardinals exhibit even more reflection. Recall one instance of this reflection for measurable cardinals κ was that if GCH holds below κ then $2^\kappa = \kappa^+$. Contrast that fact with the following proposition.

Proposition 4. *Suppose* GCH *holds below κ where κ is λ-supercompact. Then* GCH *holds below λ. In particular, if κ is supercompact and* GCH *holds below κ then* GCH *holds globally.*

Proof. Let $j : V \prec M$ witness the λ-supercompactness of κ. Because M is closed under λ-sequences we can inductively show that ${V_\alpha}^M = V_\alpha$ for $\alpha \leq \lambda + 1$. But by elementarity we have that $(2^\alpha)^M = (\alpha^+)^M$ for all $\alpha < j(\kappa)$. So then $2^\alpha \leq (2^\alpha)^M = (\beta^+)^M = \beta^+$, as desired. □

Proposition 5. *Suppose κ is supercompact. Then $V_\kappa \prec_2 V$.*[1]

To prove this we will use the following fact, which I will leave to you as an exercise.

Exercise 6. Suppose $\kappa > \omega$. Then $H_\kappa \prec_1 V$.

Proof. Consider the Σ_2 formula $\exists x \forall y \; \varphi(x, y, z)$, where φ only has bounded quantifiers, and fix $c \in V_\kappa$. For the forward direction of the implication, suppose that $V_\kappa \models \exists x \; \forall y \; \varphi(x, y, c)$, as witnessed by a. Note that $V_\kappa = H_\kappa$, because κ is inaccessible. So by the exercise we get that $\forall y \; \varphi(a, y, c)$. But then $\exists \forall y \; \varphi(x, y, c)$.

For the backward direction, suppose $\exists x \; \forall y \; \varphi(x, y, c)$, as witnessed by a. Fix λ big enough so that $a \in V_\lambda$. Now let $j : V \to M$ witness that κ is λ-supercompact. We then get, by the exercise plus elementarity, that ${V_{j(\kappa)}}^M \models \forall y \; \varphi(a, y, j(c))$. But $c \in V_\kappa$ so $j(c) = c$. So by elementarity we get that $V_\kappa \models \exists x \; \forall y \; \varphi(x, y, c)$, as desired. □

It will fall out of the combinatorial characterization of supercompactness that supercompactness does not guarantee Σ_3 reflection. And so this proposition is the best we can get.

This is as good an opportunity as any to pivot to talking about this characterization. The hint of where to begin comes from seed theory. Recall the following fact we proved back in part 1.1.

Fact 7. *Suppose $j : V \prec M$ is an ultrapower embedding and λ is an ordinal. Then ${}^\lambda M \subseteq M$ iff $j''\lambda \in M$.*

If the embedding $j : V \prec M$ witnessing that κ is λ-supercompact is an ultrapower embedding, then $j''\lambda \in M$. In fact, we don't need to assume j is an ultrapower embedding to conclude $j''\lambda \in M$, since that direction of the argument works for any embedding. Let's see what happens if we try

[1] Recall that $M \prec_2 V$ is formalizable as a single assertion in the language of ZFC.

to use $j''\lambda$ as a seed for an induced factor embedding. That is, we want to define a measure U as $X \in U$ iff $j''\lambda \in j(X)$. Where should X live? We could define U to consist of subsets $\mathcal{P}(\lambda)$, since $j''\lambda \in \mathcal{P}(j(\lambda))$. But we can refine this a bit. Note that $|j''\lambda| = \lambda < j(\kappa)$. So we can restrict U to consists of sets of subsets of λ of size $< \kappa$, and still obtain a measure.

Definition 8. Let $\kappa \leq \lambda$ be cardinals. Then $\mathcal{P}_\kappa\lambda = \{X \subseteq \lambda : |X| < \kappa\}$.

Caution! With measurable cardinals, our measure U was on κ, and so elements of U were subsets of κ, that is elements of $\mathcal{P}(\kappa)$. Here, we want our measure to be on $\mathcal{P}_\kappa\lambda$ and so elements of U will be subsets of $\mathcal{P}_\kappa\lambda$, that is elements of $\mathcal{P}(\mathcal{P}_\kappa\lambda)$.

Definition 9. Suppose $j : V \prec M$ has the property that it has critical point κ, $j(\kappa) > \lambda$, and $j''\lambda \in M$ generates all of M. Call such j a *λ-supercompactness embedding*.

By the seed lemma, λ-supercompactness embeddings are ultrapower embeddings. And by the above re-cited fact we get that ${}^\lambda M \subseteq M$ and so they really do witness λ-supercompactness.

We want to characterize the measures which give rise to λ-supercompactness embeddings.

Definition 10. A κ-complete ultrapower U on $\mathcal{P}_\kappa\lambda$ is *fine* if for each $\alpha < \lambda$ we have $X_\alpha = \{s \in \mathcal{P}_\kappa\lambda : \alpha \in s\} \in U$.

Definition 11. Suppose $f : X \to \lambda$ has domain $X \subseteq \mathcal{P}_\kappa\lambda$. Say that f is *regressive* if $f(s) \in s$ for all $s \in X$. And say that f is *regressive on* $Y \subseteq X$ if $f \upharpoonright Y$ is regressive.

Definition 12. An ultrapower U on $\mathcal{P}_\kappa\lambda$ is *normal* if every $f : \mathcal{P}_\kappa\lambda \to \lambda$ which is regressive on a set in U is constant on a set in U.

Note that both of these definitions could be made for filters.

As with ultrafilters on κ, we can characterize normality in terms of closure under diagonal intersections. To do this we need an appropriate definition of diagonal intersection for $\mathcal{P}_\kappa\lambda$.

Definition 13. Let $\langle X_i : i \in \lambda\rangle$ be a sequence of sets in $\mathcal{P}_\kappa\lambda$. Then the diagonal intersection of this sequence is

$$\mathop{\triangle}_{i\in\lambda} X_i = \left\{ s \in \mathcal{P}_\kappa\lambda : s \in \bigcap_{i \in s} X_i \right\}.$$

There is a possible notational confusion here, since we also used $\triangle_{i\in\lambda} X_i$ to denote the diagonal intersection along λ, rather than $\mathcal{P}_\kappa\lambda$. But it should be clear from context which is meant.

Exercise 14. Suppose F is a fine filter on $\mathcal{P}_\kappa\lambda$. Show that F is normal iff given any sequence $\langle X_i : i \in \lambda\rangle$ of sets from F we have that $\triangle_{i\in\lambda} X_i \in F$.

Proposition 15. *Suppose $j : V \prec M$ witnesses that κ is λ-supercompact. If U is the measure on $\mathcal{P}_\kappa\lambda$ generated by $j''\lambda$, then U is normal and fine.*

Proof. First let's check fineness. κ-completeness is free. We want to see that $j''\lambda \in j(X_\alpha)$. This follows easily by elementarity

$$j(X_\alpha) = \{s \in \mathcal{P}_{j(\kappa)} j(\lambda) : j(\alpha) \in s\}$$

plus the fact that $\alpha \in \lambda$ and so $j(\alpha) \in j''\lambda$.

To see normality, take $f : \mathcal{P}_\kappa\lambda \to \lambda$ which is regressive on a set in U. Then, $j(f)(j''\lambda) \in j''\lambda$ by the definition of U and so there is $\alpha < \lambda$ so that $j(f)(j''\lambda) = j(\alpha)$. But then $f(s) = \alpha$ on a set in U. ◻

Proposition 16. *Suppose $j : V \prec M$ is the ultrapower embedding by a normal fine measure U on $\mathcal{P}_\kappa\lambda$. Then $j''\lambda = [\mathrm{id}]_U \in M$ and ${}^\lambda M \subseteq M$.*

Proof. All we have to show is that $j''\lambda = [\mathrm{id}]_U$. Recall that $[\mathrm{id}]_U$ is a seed which generates all of M via j. First, consider $\alpha \in \lambda$. By fineness we have $X_\alpha \in U$ and so $[\mathrm{id}]_U \in j(X_\alpha)$. Therefore, $j(\alpha) \in [\mathrm{id}]_U$. Conversely, consider $\alpha \in [\mathrm{id}]_U$. Then $\alpha = [f]_U = j(f)([\mathrm{id}]_U)$ for some function $f : \mathcal{P}_\kappa\lambda \to \lambda$. Because $j(f)([\mathrm{id}]_u) \in [\mathrm{id}]_U$ we get that f is regressive on a set in U. So there is $\beta < \lambda$ so that $f(s) = \beta$ on a set in U. So $\alpha = j(f)([\mathrm{id}]_U) = j(\beta)$. That is, $\alpha \in j''\lambda$. And so we are done. □

This then gives us a ZFC-expressible characterization of when κ is supercompact: κ is supercompact iff for every $\lambda > \kappa$ there is a normal fine measure on $\mathcal{P}_\kappa\lambda$.

Now that we have this definition we can analyze it a bit more closely and see what it takes to verify whether κ is λ-supercompact. This is verified by the existence of a certain subset of $(\mathcal{P}_\kappa)$, which lives in $V_{\lambda+2}$. And all the objects we need to quantify over to check normality and fineness, namely certain functions and sets over $\mathcal{P}_\kappa\lambda$ live in $V_{\lambda+k}$ for some finite k. (Exercise: compute the minimal k that will do.)

Exercise 17. Show that the Σ_2 properties are precisely those properties witnessed by V_αs. That is, show that $\varphi(\bar{x})$ is equivalent to a formula of the form $\exists\alpha \in \mathrm{Ord}\ V_\alpha \models \psi(\bar{x})$ iff φ is Σ_2. (Hint: first show that $y = V_\alpha$ is expressible as a Π_1 assertion.)

From this exercise plus the preceding paragraph we conclude that "κ is λ-supercompact" is expressible, modulo ZFC, as a Σ_2 assertion. In fact, we can also express it as a Π_2 assertion, because every large enough V_α will witness that κ is λ-supercompact. To be more precise, we can express that κ is λ-supercompact as: "$\forall\alpha \in \mathrm{Ord}$ if $\alpha \geq \lambda + 17$ then $V_\alpha \models \exists$ a normal, fine measure on $\mathcal{P}_\kappa\lambda$".

Altogether, we get that "κ is λ-supercompact" is Δ_2^{ZFC}, in the parameters κ and λ. Therefore, "κ is supercompact" can be expressed in a Σ_3 manner.

Exercise 18. Show that if κ is the smallest supercompact cardinal then $V_\kappa \not\prec_3 V$.

Exercise 19. Show that both "there is a proper class of inaccessible cardinals" and "there is a proper class of measurable cardinals" are expressible as Π_3 assertions. Show that neither of them can be expressed in a Π_2 nor a Σ_2 manner.

The following exercise shows, on the other hand, a sense in which supercompactness can reflect supercompactness.

Exercise 20. Let $\kappa < \lambda$ and suppose κ is λ-supercompact and λ is supercompact. Show that κ is supercompact.

This exercise is most easily proven by a reflection argument. But you can also prove it directly using the normal fine measure characterization of supercompactness.

Exercise 21. Let $\kappa < \lambda$ and suppose κ is λ-supercompact and λ is supercompact. Let $\mu > \lambda$ and fix U a normal fine measure on $\mathcal{P}_\lambda\mu$. For each $x \in \mathcal{P}_\lambda\mu$ with $|x| \geq \kappa$ fix N_x a normal fine ultrafilter on $\mathcal{P}_\kappa x$. Define $W \subseteq \mathcal{P}(\mathcal{P}_\kappa\mu)$ as

$$X \in W \Leftrightarrow \{x \in \mathcal{P}_\lambda\mu : |x| \geq \kappa \text{ and } X \cap \mathcal{P}_\kappa x \in N_x\} \in U.$$

Show that W is a normal fine ultrafilter on $\mathcal{P}_\kappa\mu$.

Let us now see an alternative characterization of supercompactness.

Theorem 22 (Magidor). *κ is supercompact iff for each $\alpha > \kappa$ there is $\beta < \kappa$ and an embedding $k : V_\beta \prec V_\alpha$ so that $k(\operatorname{crit} k) = \kappa$.*

Proof. ($\Rightarrow$) Fix $\alpha > \kappa$. Let $j : V \prec M$ witness that κ is $|V_\alpha|$-supercompact. Set $\hat{\jmath} = j \upharpoonright V_\alpha$. By elementarity we get that $\hat{\jmath} : V_\alpha \prec {V_{j(\alpha)}}^M$. But M is closed under $|V_\alpha|$-sequences, so $V_\alpha = {V_\alpha}^M$ and $\hat{\jmath} \in M$. Therefore, M can see that $\hat{\jmath}$ is an elementarity and $M \models \hat{\jmath} : V_\alpha \prec V_{j(\alpha)}$. Quantifying out α, we get that $M \models \exists \xi < j(\kappa)\ \exists k : V_\xi \prec V_{j(\alpha)}$ so that $k(\operatorname{crit} k) = j(\kappa)$. Pulling this statement backward along j we get that in V there is $\xi < \kappa$ and an embedding $k : V_\xi \prec V_\alpha$ so that $k(\operatorname{crit} k) = \kappa$. So ξ is the desired β.

($\Leftarrow$) Fix $\lambda > \kappa$. Take $\beta < \kappa$ and $k : V_\beta \prec V_{\lambda+\omega}$ so that if $\delta = \operatorname{crit} k$ then $k(\delta)\kappa$. In then follows that $\beta = \gamma + \omega$ where $k(\gamma) = \lambda$. Now observe that $\mathcal{P}(\mathcal{P}_\delta \gamma) \subseteq V_\beta$ and that $k''\beta \in \mathcal{P}_\kappa \lambda$. Define $U \subseteq \mathcal{P}(\mathcal{P}_\delta \beta)$ as

$$X \in U \Leftrightarrow k''\gamma \in k(X).$$

By the usual argument, U is a normal ultrafilter over $\mathcal{P}_\delta \beta$. But also $U \in V_\beta$ and so by elementarity $k(U)$ is a normal ultrafilter over $\mathcal{P}_{j(\delta)} j(\beta) = \mathcal{P}_\kappa \lambda$. So κ is λ-supercompact. □

As an application of supercompactness, let us see that supercompact cardinals admit functions that "guess" any object in the universe via a supercompactness embedding.

Theorem 23 (Laver). *Suppose κ is supercompact. Then there is a partial function $\ell \,\vdots\, \kappa \to V_\kappa$ so that for any x and any cardinal θ with $x \in H_{\theta^+}$ there is a θ-supercompactness embedding $j : V \prec M$ with $j(\ell)(\kappa) = x$. Such an ℓ is called a* Laver function *for the supercompact cardinal κ.*

Proof. We build ℓ by transfinite recursion. We suppose $\ell \upharpoonright \alpha$ has already been defined and we want to define $\ell(\alpha)$. We have two cases to consider. The first is that there is some λ and some $x \in H_{\lambda^+}$ so that x is not guessed by $\ell \upharpoonright \alpha$ for any λ-supercompactness embedding for α. That is, there is no λ-supercompactness embedding $h : V \prec M$ with critical point α so that $h(\ell \upharpoonright \alpha)(\alpha) = x$. In this case, let λ be least such this happens and pick any x for this λ and set $\ell(\alpha) = x$. If there is no such λ and x, then we leave $\ell(\alpha)$ undefined.

We inductively do this for all $\alpha < \kappa$ to produce $\ell \,\vdots\, \kappa \to V$. I claim that $\ell \,\vdots\, \kappa \to V_\kappa$. That is, we have to see that if there is λ with $x \in H_{\lambda^+}$ so that no λ-supercompactness embedding $h : V \prec M$ with critical point $\alpha < \kappa$ has $h(\ell \upharpoonright \alpha)(\alpha) = x$, then there is such a $\lambda < \kappa$. But observe that this property of λ is Π_2 with parameters from V_κ: α being λ-supercompact is Δ_2, and checking whether $h(\ell \upharpoonright \alpha)(\alpha) = x$ is Σ_0 in parameters x, $\ell \upharpoonright \alpha$, and α, and so saying there is $x \in H_{\lambda^+}$ with no λ-supercompactness embedding with $h(\ell \upharpoonright \alpha)(\alpha) = x$ is Π_2. So because $V_\kappa \prec_2 V$ there must be such a $\lambda \in V_\kappa$.

Now let us check that ℓ really is a Laver function. Suppose toward a contradiction that it is not, and let θ be a minimal failure. That is, there is $x \in H_{\theta^+}$ so that $x \neq h(\ell)(\kappa)$ for any θ-supercompactness embedding $h : V \prec N$ with critical point κ, but this is not true for any cardinal $< \theta$. Let $j : V \prec M$ be a $(2^\theta)^{<\kappa}$-supercompactness embedding with critical point κ. Then M is sufficiently closed so that it has all the supercompactness measures on $\mathcal{P}_\kappa \theta$ and all functions $\mathcal{P}_\kappa \theta \to V_\kappa$ as are in V. Thus, M must agree with V that x is not guessed by ℓ for any θ-supercompactness embedding. More, M agrees that θ is minimal so that this happens. Now use that $\ell = j(\ell) \upharpoonright \kappa$, and get that $j(\ell)(\kappa)$ is defined and so $y = j(\ell)(\kappa) \in H_{\theta^+}$ is not guessed by ℓ with respect to any θ-supercompactness embedding.

Consider the embedding $j_0 : V \prec M_0$ induced by the seed $j''\theta$ using j. This is a *theta*-supercompactness embedding and this M_0 is the collapse of the seed hull of $j''\theta$. So by composing

j with k, the inverse of the collapse map, we get $j = k \circ j_0$. Note that y is in this seed hull and, since each ordinal $< \theta$ is in this seed hull, the seed hull correct computes the elements of H_{θ^+}, we get that $k(y) = y$. In particular, it must be that $y \in M_0$. So

$$y = j(\ell)(\kappa) = k(j_0(\ell)(\kappa)) = j_0(\ell)(\kappa).$$

That is, we have seen that ℓ guesses y with respect to the θ-supercompactness embedding j_0. But M sees the measure which gives rise to j_0, and M has sufficient closure to compute $j_0(\ell)(\kappa) = y$. This contradicts that y was not guessed in M by ℓ via any θ-supercompactness embedding. □

Other kinds of guessing principles have been studied by set theorists. One prominent one is Jensen's diamond principle, which the following exercises have you look at a bit.

Definition 24. Let κ be an infinite cardinal. Then $\diamondsuit_\kappa$ asserts that there is a sequence $\langle D_\alpha : \alpha < \kappa \rangle$ so that for any $X \subseteq \kappa$ the set $\{\alpha < \kappa : X \cap \alpha = D_\alpha\}$ is stationary. In particular, it must be that $D_\alpha \subseteq \alpha$ stationarily often.

Exercise 25. Show that if κ is measurable then $\diamondsuit_\kappa$ holds. (Hint: use the construction of a Laver function as inspiration for your construction of a diamond sequence for κ.)

Exercise 26. Show that $\diamondsuit_{\omega_1}$ implies the continuum hypothesis. More generally, show that $\diamondsuit_{\kappa^+}$ implies $2^\kappa = \kappa^+$.

In fact, $\diamondsuit_{\omega_1}$ is strictly stronger than CH, but proving this would require tools beyond our current reach. One might wonder whether $\diamondsuit_\kappa$ is even consistent in general. Jensen showed that Gödel's constructible universe, the minimum inner model, satisfies $\diamondsuit_\kappa$ for every uncountable regular κ.

Exercise 27. Show that $\diamondsuit_{\omega_1}$ implies there exists an ω_1-Suslin tree, where a κ-Suslin tree is a tree of height κ so that each branch and anti-chain of the tree has cardinality $< \kappa$.

$\diamondsuit$ principles can be used for constructions, building up an object of size κ by smaller subobjects, using the diamond sequence to anticipate future obstacles. The most notable example of this is Shelah's proof that $\diamondsuit_\kappa$ for all uncountable regular κ implies that every Whitehead group is free, thus showing that Whitehead's problem from abstract algebra consistently with ZFC has a positive answer. (Shelah also showed that Whitehead's problem consistently has a negative answer, so it is independent.)

2. An upper limit

The previous section suggests an ultimate limit to the embedding-based large cardinals. If we get stronger and stronger principles by requiring the target model to contain more and more information, then why not go all the way and ask for κ to be the critical point of an embedding $j : V \prec V$?

There's an obstacle to formalizing this.

Proposition 28. *The only definable elementary embedding $j : V \prec V$ is the identity.*

Proof. First let's do the case where j is definable without parameters. Suppose $j : V \prec V$ is not the identity. Let $\kappa = \operatorname{crit}(j)$. Note that κ is definable without parameters, being the critical point of j. Let $\varphi(x)$ be the formula which defines κ—that is, $\varphi(x)$ holds iff $x = \kappa$. Then, by elementarity, $\varphi(x)^V \Leftrightarrow \varphi(j(x))^V$. Note that on both sides of this we are evaluating in V, since V is both the target and the domain of j. So $\varphi(\kappa)$ holds iff $\varphi(j(\kappa))$ holds, contradicting that κ is the unique witness of φ.

In other words, what we just showed is that if x is definable without parameters and $j : V \prec V$, then $j(x) = x$.

Now let's do the case j is defined using a parameter. That is, there is a set p and a formula $\gamma(x, y, z)$ so that $j_p(x) = y$ iff $\gamma(x, y, p)$ is an elementary embedding from V to V. Suppose towards a contradiction that j_p is not always the identity. Let α be the minimal rank of a parameter p so that j_p is not the identity. And let κ be the smallest critical point of $j_p \neq \mathrm{id}$ for p of rank α. Then κ is definable without parameters. So by elementarity we get that $j_p(\kappa) = \kappa$ for all p of rank α. But this has to be false for at least one p. Contradiction. □

So it would not be possible to talk about such embeddings in our context where classes are just a way of speaking about certain formulae. With this obstacle in mind, we make the following definition.

Definition 29. κ is a *Reinhardt* cardinal if κ is the critical point of an embedding $j : V_\mu \prec V_\mu$ with $\mu > \kappa$ inaccessible.

This definition can be sensibly formalized within ZFC, since all the objects involved are sets. The idea is, we think of V_λ as an approximation to the full universe of sets. But since it is only an initial segment of the universe, we can look on it from above and see that the "classes" from V_μ's point of view are just sets in $V_{\mu+1}$.

Nevertheless, Reinhardt cardinals are known to not exist.

Theorem 30 (Kunen)**.** ZFC *proves there are no Reinhardt cardinals.*

We will use the following ZFC theorem. Here, $\mathcal{P}_{=\kappa} A$ is the collection of subsets of A of cardinality κ.

Theorem 31 (Erdős–Hajnal, over ZFC)**.** *Every infinite cardinal λ has an ω-*Jónsson function*. That is, there is a function $f : \mathcal{P}_{=\omega}\lambda \to \lambda$ so that for all $y \in \mathcal{P}_{=\lambda}\lambda$ we have $f''\mathcal{P}_{=\omega} Y = \lambda$.*

Proof. Consider the equivalence relation $\sim$ on $\mathcal{P}_{=\omega}\lambda$ defined as $x \sim y$ if they agree on a tail—that is there is $\alpha < \sup x$ so that $x \setminus \alpha = y \setminus \alpha$. Using AC, pick $x_E \in E$ for each $\sim$-equivalence class E. For $y \in \mathcal{P}_{=\omega}\lambda$, let E_y be the equivalence class for y. Now given $y \in \mathcal{P}_{=\omega}\lambda$ set $g(y)$ to be the least $\alpha \in x_{E_y}$ so that $y \setminus (\alpha + 1) = x_{E_y} \setminus (\alpha + 1)$. That is, α is the least witness that $y \sim x_{E_y}$.

It now suffices to find $A \in \mathcal{P}_{=\lambda}\lambda$ so that every $B \in \mathcal{P}_{=\lambda} A$ has $G''\mathcal{P}_{=\omega} B \supseteq A$. If there is such an A, then because it is in bijection with λ we can use that bijection and g to define the desired f.

Suppose toward a contradiction there is no such A. Inductively, for each $n \in \omega$, find $A_n \in \mathcal{P}_{=\lambda}\lambda$ and $\alpha_n \in \lambda$ so that

- $A_n \supseteq A_{n+1}$;
- $\alpha_{n+1} \in A_n \setminus (\alpha_n + 1)$; and
- $\alpha_{n+1} \notin g''\mathcal{P}_{=\omega}A_{n+1}$..

We can continue the induction for each step by our assumption there is no such A. Now set $y = \{a_n : n \in \omega\}$. Pick m so that for some $\alpha \in \sup x_{E_y}$ we have $\{\alpha_n : n \geq m\} = x_{E_y} \setminus \alpha$ Then $g(\{\alpha_n : n \geq m\}) = \alpha_m$ and so $\alpha_m \in g''\mathcal{P}_{=\omega}A_m$. This contradicts the construction of A_m. □

Proof of Kunen's inconsistency theorem. Suppose toward a contradiction that $j : V_\mu \prec M \subseteq V_\mu$ is an elementarity embedding with critical point κ. Set $\kappa_0 = \kappa$ and $\kappa_{n+1} = j(\kappa_n)$. Let $\lambda = \sup_n \kappa_n$. Observe that

$$j(\lambda) = j(\sup_n \kappa_n) = \sup_n j(\kappa_n) = \sup_n \kappa_{n+1} = \lambda.$$

Indeed, λ is the least ordinal $> \kappa$ which is fixed by j, because if $\kappa < \alpha < \lambda$ then $\kappa_n < \alpha \leq \kappa_{n+1}$ and so $j(\kappa_n) = \kappa_{n+1} < j(\alpha)$.

Let us now see that $j''\lambda \notin M$ and thus $M \neq V_\mu$. Suppose otherwise toward a contradiction. By the Erdős–Hajnal theorem pick $f : \mathcal{P}_{=\omega}\lambda \to \lambda$ so that for all $y \in \mathcal{P}_{=\lambda}\lambda$ we have $f''\mathcal{P}_{=\omega}y = \lambda$. By elementarity and using that $j(\lambda) = \lambda$, we get that $j(f) : \mathcal{P}_{=\omega}\lambda \to \lambda$ is a function so that for all $y \in M \cap \mathcal{P}_{=\lambda}\lambda$ we have $j(f)''\mathcal{P}_{=\omega}y = \lambda$. In particular, since we get that $j(f)''\mathcal{P}_{=\omega}j''\lambda = \lambda$.

I claim that $j(f)''\mathcal{P}_{=\omega}j''\lambda \subseteq j''\lambda$ which would then imply that $j''\lambda = \lambda$, which would be a contradiction. To see this, pick $s \in \mathcal{P}_{=\omega}j''\lambda$. Then there is $t \in \mathcal{P}_{=\omega}\lambda$ so that $j(t) = j''t = s$. But then

$$j(f)(s) = j(f)(j(t)) = j(f(t)) \in j''\lambda.$$ □

A couple remarks are in order. First, the axiom of choice was used essentially in the proof of the Erdős–Hajnal theorem. A natural question is whether Reinhardt cardinals can be proved nonexistent just from ZF. So far, no one has succeeded in doing so...

We can also ask how far down Kunen's argument can be pushed. It's straightforward to see that his argument rules out an elementary embedding $j : V_{\lambda+2} \to V_{\lambda+2}$, where $\lambda = \sup j^n(\kappa)$, since ω-Jónsson function lives in $V_{\lambda+2}$. In other words, our requirement that $\mu > \kappa$ be inaccessible in the definition of Reinhardt cardinals was overkill for what's needed for Kunen's theorem. But the argument does not work if we lower the index. And so far no one has found an alternative argument which works for these lower indices. Instead, set theorists have studied the *rank-into-rank* cardinals, those λ so that there is an elementary embedding $j : V_\lambda \prec V_\lambda$.

(Kameryn J. Williams) University of Hawaiʻi at Mānoa, Department of Mathematics, 2565 McCarthy Mall, Keller 401A, Honolulu, HI 96822, USA

E-mail address: kamerynw@hawaii.edu

URL: http://kamerynjw.net