MATH655 LECTURE NOTES: PART 1.0 INACCESSIBLE CARDINALS

KAMERYN J. WILLIAMS

This part of the course is about large cardinals, those cardinal numbers which exceed ZFC in logical strength. They give a hierarchy of principles which can be used to measure the strength of principles which exceed the standard axioms.

Why should we believe in the existence, or even just the consistency of large cardinals, if they go beyond ZFC? That is an excellent question, which we can only get back to after having spent some time studying large cardinals. For now, we will take it as a given that all of the principles we are studying are consistent. (With the exception of Reinhardt cardinals, which we will see contradict ZFC.)

For the first section of Part 1 we will cover inaccessible cardinals, near the weakest of the large cardinals.

1. Inaccessible cardinals, a beginning

Let us begin with a useful bit of notation.

Definition 1. Let κ, λ be cardinals. Then

$$\kappa^{<\lambda} = \sup_{\mu<\lambda} \kappa^\mu.$$

Definition 2 (Hausdorff)**.** An uncountable cardinal κ is *inaccessible* if it is a regular strong limit. That is, $\operatorname{cof} \kappa = \kappa$ and $2^{<\kappa} = \kappa$.

Proposition 3. *Suppose κ is inaccessible. Then for $x \subseteq V_\kappa$ we have $x \in V_\kappa$ iff $|x| < \kappa$.*

Proof. ($\Rightarrow$) It is enough to show that $|V_\alpha| < \kappa$ for $\alpha < \kappa$. This can be proven by induction on α, using that $2^{<\kappa} = \kappa$.

($\Leftarrow$) Let $x \subseteq V_\kappa$ have $|x| < \kappa$. Because κ is regular, there is $\alpha < \kappa$ so that $\operatorname{rank} y < \alpha$ for all $y \in x$. But then $x \in V_{\alpha+1} \subseteq V_\kappa$. □

Theorem 4. *If κ is inaccessible then $V_\kappa \models \mathsf{ZFC}$.*

Proof. We have seen that all the axioms except Replacement already hold. So let's check Replacement. Suppose $x \in V_\kappa$ and $F : x \to V_\kappa$ is a function. Then $|F''x| \leq |x| < \kappa$. So by the previous proposition, $F''x \in V_\kappa$. □

As a consequence of this theorem we can immediately draw two conclusions. First, ZFC does not prove the existence of inaccessible cardinals. This is because if κ is the least inaccessible cardinal then $V_\kappa \models \mathsf{ZFC}$ + "there are no inaccessible cardinals". Moreover, ZFC + "there is an inaccessible" cardinal proves Con(ZFC), where Con(ZFC) is the formal statement asserting that ZFC is consistent. This is just because if there is an inaccessible cardinal then there is a model of ZFC and if a theory has a model then it is consistent.

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We summarize this second fact by saying that ZFC + "there is an inaccessible" exceeds ZFC in consistency strength. In general, if S and T are two set theories, we say that S exceeds T in consistency strength if S proves $\mathrm{Con}(\mathsf{ZFC})$. Thinking in terms of models: given a model of S we can produce a model of T, but there is not a reverse process.[1]

We will treat ZFC as the base theory for our investigations, implicitly including it in our stronger theories. For example, we will simply say that inaccessibles are stronger than ZFC in consistency strength to mean the theory ZFC + "there is an inaccessible cardinal" proves $\mathrm{Con}(\mathsf{ZFC})$.

Assertions like "there is an inaccessible cardinal" as called *large cardinal assertions*. This is not a formal term, but rather a know-it-when-you-see-it sort of thing. In general, a large cardinal assertion asserts the existence of a cardinal κ satisfying some property so that ZFC cannot prove such κ exists. So the large cardinals are large in the sense of consistency strength. And for most of them, the largeness is also in terms of cardinality. We haven't yet seen other large cardinal notions, but we will later see that *measurable* cardinals exceed inaccessibles in consistency strength. This is because if κ is measurable there are lots and lots of inaccessible cardinals $< \kappa$.

Once we have one large cardinal notion, we get stronger assertions by asserting the existence of multiple cardinals of that type. For example, 2 inaccessibles are stronger than 1 inaccessible, because if κ is the second inaccessible then $V_\kappa \models \mathsf{ZFC}$ + "there is 1 inaccessible". In general, if κ is the $(\alpha+1)$-th inaccessible then $V_\kappa \models \mathsf{ZFC}$ + "there are α many inaccessibles". So there is a corresponding hierarchy of theories extending ZFC in consistency strength by asserting more and more inaccessibles.

A bit more interesting is the following. Say that κ is 2-*inaccessible* if κ is inaccessible and a limit of inaccessible cardinals.

Note that if κ is 2-inaccessible then $V_\kappa \models \mathsf{ZFC}$ + "there are Ord many inaccessibles". So 2-inaccessibles are stronger than having multiple inaccessibles. And this is just the second step in a hierarchy of larger and larger cardinals.

Definition 5. Define the *α-inaccessible cardinals* by induction on α.

- κ is 0-inaccessible if κ is regular.
- κ is $(\alpha+1)$-inaccessible if κ is in inaccessible limit of α-inaccessible cardinals.
- For γ limit, κ is γ-inaccessible if κ is α-inaccessible for all $\alpha < \gamma$.

Observe that 1-inaccessibility is the same as inaccessibility. And an easy induction shows that if $\beta < \alpha$ then an α-inaccessible must also be β-inaccessible. (Exercise: do it!)

Exercise 6. Show that if $\kappa < \alpha$ then κ cannot be α-inaccessible.

An alternative definition of the α-inaccessibles goes through the following operation on ordinals. Let Reg denote the class of regular cardinals. If X is a class of cardinals, then κ is a *limit point* of X if κ is the supremum of a set of cardinals from X. Given a class X of cardinals, let

$$I(X) = \{\alpha \in X : \alpha \text{ is an inaccessible limit point of } X\}.$$

[1]Massive caution here: this is an informal assertion meant to give some intuition, not a formal theorem. An issue with trying to turn this intuition into a theorem is that you have cash out what a process is in this context. To illustrate the problem: suppose that there is a model M of ZFC + "there is an inaccessible cardinal". Now consider the following 'process' to produce a model of ZFC + "there is an inaccessible cardinal" from a model of ZFC: given $N \models \mathsf{ZFC}$, throw it away then output M. Your formal definition should exclude silly things like this.

We can iterate this operation:

$$\begin{aligned} I^0(X) &= X \\ I^{\alpha+1}(X) &= I(I^\alpha(X)) \\ I^\lambda(X) &= \bigcap_{\alpha<\lambda} I^\alpha(X) \qquad (\lambda \text{ limit}) \end{aligned}$$

Then $I^\alpha(\mathrm{Reg})$ is the collection of α-inaccessible cardinals.

Say that κ is *hyper-inaccessible* if κ is κ-inaccessible. These may seem like unfathomably large cardinals that surely must be the boundary of consistency but they are actually rather far down in the large cardinal hierarchy.

Exercise 7. Say that a cardinal κ is *Mahlo* if the set of inaccessible cardinals $< \kappa$ is stationary in κ.[2] Show that if κ is Mahlo then it is hyper-inaccessible.[3] (Hint: Start by showing that the limit points $< \kappa$ of $I^\alpha(\mathrm{Reg} \cap \kappa)$ form a club subset of κ for all $\alpha < \kappa$. Use this to show that κ itself is hyper-inaccessible.)

We have not yet seen much of the large cardinal hierarchy, but Mahlo cardinals are low in it.

[2]Recall that $S \subseteq \kappa$ if S intersects every club subset of κ, where $C \subseteq \kappa$ is club if it is closed and unbounded in κ.

[3]In fact, κ is a limit of hyper-inaccessible cardinals. As a more challenging exercise, try to prove this. (Hint: you need to consider diagonal intersections of clubs. If $\langle C_\alpha : \alpha < \kappa \rangle$ is a sequence of subsets of κ, then their diagonal intersection is

$$\mathop{\triangle}_{\alpha<\kappa} C_\alpha = \{\alpha < \kappa : \alpha \in \bigcap_{\beta<\alpha} C_\alpha\}.$$

Show that if the C_α are all club then so is $C = \triangle_{\alpha<\kappa} C_\alpha$.)

2. Going downward from inaccessibles: transitive models of ZFC

Given an inaccessible cardinal κ we have that V_κ is a model of ZFC. Let us apply the downward Löwenheim–Skolem theorem to this structure. We get that for every infinite cardinal $\lambda < \kappa$ there is $(\bar{M}_\lambda, E_\lambda) \prec (V_\kappa, \in)$. Since each E_λ must be well-founded—being the restriction of $\in$ to $\bar{M}_\lambda$—and extensional, by the Mostowski collapse lemma there are transitive sets M_λ so that $(M_\lambda, \in) \cong (\bar{M}_\lambda, \in)$. So there are transitive sets M_λ of every infinite cardinality $\lambda < \kappa$ which elementarily embed into V_κ.[4] In particular, there are transitive models of ZFC of every infinite cardinality $\leq \kappa$.

Exercise 8. Suppose M is a transitive set of cardinality λ. Show that every $x \in M$ has cardinality $\leq \lambda$.

What up with that? I thought ZFC asserts that every set has a powerset, and that Cantor's theorem implies that A and $\mathcal{P}(A)$ have different cardinalities. But if M is a countable transitive model of ZFC then it has to contain both $\mathbb{N}$ and $\mathbb{R}$, so how can they both be countable?

This mystery is known as the *Skolem paradox*.[5] To solve it we will need to look more carefully at the logical structure of the assertion $|A| < |\mathcal{P}(A)|$. And this is a good excuse to consider the general phenomenon.

Definition 9. A formula $\varphi(\bar{x})$ is called Δ_0 if the only quantifiers in φ are *bounded*, that is of the form $\exists y \in z$ or $\forall y \in z$.[6]

For the logicians in the room: this convention differs from that in model theory, where the Δ_0 formulae are the quantifier-free ones. In this context, this notion is too weak—you can say basically nothing about set theory with just boolean operations. This is the right basic notion for this context, as the proposition below illustrates.

Proposition 10. *Let M be a transitive set, and suppose $\bar{a} \in M$. Let $\varphi(\bar{x})$ be Δ_0. Then $M \models \varphi(\bar{a})$ iff $\varphi(\bar{a})$.*

This phenomenon is expressed by saying: Δ_0 properties are absolute for transitive models*, where we often leave "for transitive models" implicit.*

Proof. This proved by induction on formulae. The atomic case is immediate, because the membership relation for M is the true $\in$. The boolean cases are simple. So let us consider the bounded quantifier step in the induction. That is, we assume that for all $b, \bar{a}$ in M that $M \models \varphi(b, \bar{a})$ iff $\varphi(b, \bar{a})$. We want to show that $M \models \exists x \in c\ \varphi(x, \bar{a})$ iff $\exists x \in c\ \varphi(x, \bar{a})$. For the forward direction, suppose $b \in c$ witnesses that $M \models \exists x \in c\ \varphi(x, \bar{a})$. Then b witnesses $\exists x \in c\ \varphi(x, \bar{a})$, using the inductive hypothesis. The other direction is similar, using that M is transitive to know that $b \in c$ is in M. $\square$

Exercise 11. Show that all of the following properties can be expressed with Δ_0 formulae, and so they are all absolute for transitive models.

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[4] An *elementary embedding* $j : M \to N$ is an embedding so $j''M \prec N$.

[5] The Skolem paradox is not a paradox in the sense of Russell's paradox, where we get a logical contradiction. Rather, it's a paradox in that it's a counterintuitive result we need to explain.

My favorite paradox of this flavor is what Wikipedia calls the potato paradox: You have 100 pounts of potatoes, which are 99% water by weight. You leave them out and they dehydrate until they are only 98% water by weight. How much do they weigh now? (The answer: 50 pounds. Whoa! (Exercise: explain this 'paradox'.))

[6] These can be treated as abbreviations for $\exists y\ (y \in z \wedge \ldots)$ and $\forall y\ (y \in z \Rightarrow \ldots)$.

- x is an ordered pair.
- R is a relation.
- f is a function.
- $y = f(x)$.
- $g = f \upharpoonright x$.
- $B = f''A$.
- f is a one-to-one function.
- f is onto B.
- f is a bijection from A to B.
- $x \subseteq y$.
- $x = y \cup z$.
- $x = y \cap z$.
- $x = \emptyset$.
- $x = y \times z$.
- x is transitive.
- x is an ordinal.
- x is a limit ordinal.
- x is a successor ordinal.

Starting from the Δ_0 formulae, we can build upward to get a hierarchy which captures all formulae in the language of set theory.

Definition 12. The *Lévy hierarchy* is defined as follows.

- The Δ_0 formulae are those with only bounded quantifiers. These are also called Σ_0 and Π_0.
- A formula is Σ_{n+1} if it is of the form $\exists \bar{x} \varphi$ where φ is Π_n.
- A formula is Π_{n+1} if it is of the form $\forall \bar{x} \varphi$ where φ is Σ_n.

Let us first verify that this hierarchy does indeed capture all formulae.

Proposition 13. *Every formula in the language of set theory is equivalent, over* ZFC*, to one in the Lévy hierarchy.*

Proof. Proved by induction on formulae. The only substantive case is showing that bounded quantifiers can be pulled inward. That is, we want to show that every formula of the form $\forall x \in a\ \exists y\ \varphi(x, y, \bar{c})$ is equivalent to a formula of the form $\exists b\ \forall x \in a\ \exists y \in b\ \varphi(x, y, \bar{c})$.[7] The backward direction of the implication is immediate. Let us see that the forward direction is given by the Replacement axiom. So suppose that for each $x \in a$ there is y so that $\varphi(x, y, \bar{c})$. Then by Replacement there is some α so that for each $x \in a$ there is $y \in V_\alpha$ so that $\varphi(x, y, \bar{c})$. In other words, V_α is our desired b. □

Remark 14. The axiom schema

$$[\forall x \in a\ \exists y\ \varphi(x, y, \bar{c})] \Rightarrow [\exists b\ \forall x \in a\ \exists y \in b\ \varphi(x, y, \bar{c})]$$

is known as the Collection schema. We have just seen that, over the other axioms of ZF, that the Replacement schema implies the Collection schema. And the other direction of implication is immediate, so the two schemata are equivalent, over the other axioms.

Project Idea 15. However, the axiom of Powerset is essential in proving the equivalence. In the absence of this axiom, Collection is strictly stronger than Replacement. A possible project is to investigate set theory without the axiom of Powerset.

Let us return to our mystery. The following are a pair of easy observations, using the absoluteness of Δ_0 properties.

Exercise 16. Show that Σ_1 properties are *upward absolute* while Π_1 properties are downward absolute. That is, if transitive $M \models \varphi(\bar{a})$ where φ is Σ_1, then $\varphi(\bar{a})$. And if $\varphi(\bar{a})$ holds where φ is Π_1 and $\bar{a}$ are in transitive M, then $M \models \varphi(\bar{a})$.

[7] It's not necessary to also check the dual fact, using standard rules for quantifiers and negation.

Let us think how to formally write down "there is no bijection from A to B". We know that "f is a bijection from A to B" is Δ_0, so this can be written in the form $\neg\exists f\ \varphi(f, A, B)$ where φ is an appropriate Δ_0 formula. So by standard quantifier rules this can be written as $\forall f\ \neg\varphi(f, A, B)$. That is, it can be expressed as a Π_1 formula.

So we know that two sets having different cardinalities is downward absolute; if A and B have different cardinalities (in V), then no transitive M can think they have the same cardinality. The Skolem paradox shos that the absoluteness does not go in the other direction, and so there is no Σ_1 way to express that two sets have different cardinalities. Let $M \models \mathsf{ZFC}$ is countable and transitive, and suppose $a, b \in M$ are two sets which M thinks have different cardinalities. Since both a and b are countable (in V), there is a bijection between them. But M doesn't see that bijection; it's missing too many sets.

This explains the mystery of the Skolem paradox. If $M \models \mathsf{ZFC}$ is countable and transitive, then it has sets it thinks are ω and $\mathcal{P}(\omega)$. It is correct about which set is ω, but it is not correct about which set is $\mathcal{P}(\omega)$. Instead, $\mathcal{P}(\omega)^M$, what M thinks is the powerset of ω, is seen externally to be some countable collection of subsets of ω. But M does not see the bijection witnessing that it is countable, so it thinks that it is uncountable!

It gets better. Let κ be the second inaccessible cardinal. Then there is countable transitive M which elementarily embeds into V_κ. Since $V_\kappa \models$ "there is an inaccessible cardinal", there is an ordinal $\alpha \in M$ so that M thinks α is an inaccessible cardinal. But externally, we can see that what M thinks is a large cardinal is in fact some countable ordinal!

In short, models of set theory can be very, very, very wrong about cardinality. Indeed, in part 2 we will see how to expand the universe of sets to a larger universe, still satisfying ZFC. Moving to such a "forcing extension" can collapse cardinals; it could be that κ is an uncountable cardinal in V but in this larger forcing extension κ is a countable ordinal! Everything is relative; there is no truth; the government did 9/11.

Having resolved the mystery, let us round up some positive absoluteness results, showing that certain properties are absolute. That is to say, not everything is relative. We begin with perhaps the most important absoluteness result.

Theorem 17. *Well-foundedness is absolute for transitive models of* ZFC*. That is, if* $M \models \mathsf{ZFC}$ *is transitive and* $R \in M$ *is a binary relation then* $M \models R$ *is well-founded iff* R *really is well-founded.*

Proof. It is enough to see that "R is well-founded" has both a Σ_1 and a Π_1 characterization. For the Π_1 characterization, simply note that the definition—every nonempty subset of the domain has a minimal element—is already Π_1.

For the Σ_1 characterization, we will see that well-foundedness is characterized by the existence of a ranking function. Say that a function ρ from the domain of a relation R to the ordinals is a *ranking function* if $x\,R\,y$ implies $\rho(x) < \rho(y)$. If a relation R admits a ranking function, then it must be well-founded; otherwise, if $D \subseteq \operatorname{dom} R$ lacks a minimal element then $\rho'' D$ lacks a least element, which is impossible. For the other direction, we build a ranking function for well-founded R by transfinite recursion. Namely, recursively define $\rho(x) = \sup\{\rho(y) + 1 : y\,R\,x\}$. Then this gives a ranking function. Finally, observe that "ρ is a ranking function for R" is Σ_1. □

A couple remarks on this argument. First, note that we didn't need the full strength of ZFC to prove that well-foundedness is equivalent to having a ranking function. In particular, we didn't use the Powerset axiom. On the other hand, we did need some strength to carry out the transfinite recursion argument. Indeed, well-foundedness is not absolute for transitive sets, without any assumption on what axioms they satisfy. (Exercise: find a transitive set t with a binary relation $R \in t$ so that $t \models R$ is well-founded, but R is actually ill-founded.)

Second, note that we used essentially the fact that being an ordinal is a Δ_0 property to get that having a ranking function is Σ_1. That being an ordinal is Δ_0 comes from the von Neumann definition, plus the Foundation axiom. This is possibly the main reason why set theorists prefer the definition of ordinals we do; the absoluteness of well-foundedness is super important, and this definition lets us prove that.

Definition 18. Properties which have both Σ_1 and Π_1 characterizations are known as Δ_1. Strictly speaking, we should really write Δ_1^{ZFC}, since we used ZFC to prove the two characterizations are equivalent. More generally, if we used some other base theory T we would call them Δ_1^T. Then Δ_1^T properties are absolute between transitive models of T.

The next couple exercises give some more absoluteness results.

Exercise 19. Show that ordinal addition, multiplication, and exponentiation are absolute between transitive models of ZFC.[8]

Exercise 20. Show that the relation $A \models T$ is absolute for transitive models of ZFC (without Powerset). That is, if $M \models \mathsf{ZFC}$ (minus Powerset) is transitive, A is a structure in M, and $T \in M$ is a theory in the language of A, then $M \models [A \models T]$ iff $A \models T$. (Hint: you want to find Σ_1 and Π_1 characterizations for the satisfaction relation $A \models T$.)

The next two exercises are about the existence of transitive models of ZFC and the consistency of ZFC. Note that the arguments don't use much about ZFC, and the same results hold if ZFC is replaced by another reasonable set theory, e.g. one asserting the existence of large cardinals.

Exercise 21. Show that (under the assumption that there is a transitive model of ZFC) there is a transitive model M of ZFC so that $M \models$ "there is no transitive model of ZFC".[9]

On the other hand, $\mathrm{Con}(\mathsf{ZFC})$ is an arithmetic statement, since syntactic statements about formulae, proofs, etc. can be coded as statements about natural numbers.[10] Since all transitive models of ZFC contain ω and the arithmetic operations on ω are absolute, this means that arithmetic operations are absolute for transitive models of (a fragment of) ZFC. Thus, if there is transitive $M \models \mathsf{ZFC}$, then $\mathrm{Con}(\mathsf{ZFC})$ is true and so $M \models \mathrm{Con}(\mathsf{ZFC})$.

Exercise 22. Explain this. If M is a transitive model of ZFC so that $M \models$ "there is no transitive model of ZFC", how can $M \models \mathrm{Con}(\mathsf{ZFC})$?

This last exercise relates the set theorist's Δ_0 to the model theorist's Δ_0 by expanding the language used.

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[8] Indeed, you need much less; the weak set theory KP suffices.

[9] In fact, more is true. There is a transitive model $M \models \mathsf{ZFC}$ so that no transitive $N \subsetneq M$ is a model of ZFC. But proving this stronger fact would require going through Gödel's constructible universe, which we will not cover in this class.

[10] The original argument for this is due to Gödel. But to anyone who has used a modern computer this is clear. All kinds of things can be represented as really long binary strings, which can be thought of as numbers written in binary. So one just has to believe that the operations computers do on binary strings can be cast as arithmetic operations on those numbers.

Exercise 23. For each Δ_0 formula $\varphi(\bar{x})$ of arity n, let R_φ be a corresponding relation symbol of arity n. Expand ZFC in this extended language by adding the following axioms for each Δ_0 formula φ:

$$\forall \bar{x}\ \varphi(\bar{x}) \Leftrightarrow R_\varphi(\bar{x}).$$

Call this expanded theory ZFC^+. Show that ZFC^+ is a conservative expansion of ZFC. (That is, any theorem of ZFC^+ which is in the more limited the language of ZFC with just $\in$, is already a theorem of ZFC.) Show that over ZFC^+, every Σ_k (respectively, Π_k) formula in the language with just $\in$ is equivalent to Σ_k (respectively, Π_k) formula in the expanded language without any bounded quantifiers.

3. Back to large cardinals

Having explored countable transitive models, let us return to large cardinals. First, let us see that the inaccessible cardinals enjoy a nice characterization based on second-order logic. In part $\frac{1}{2}$ a couple exercises were about formulating axioms for $\mathbb{R}$ and $\mathbb{N}$ in second-order logic. We can also formulate set theory in second-order logic. Here, I will use the convention that lowercase variables refer to first-order objects, i.e. those in the domain of discourse, while uppercase variables refer to second-order objects, i.e. subsets of the domain of discourse.

Definition 24. ZFC_2 is the formulation of ZFC in second-order logic. Specifically all axioms except the Separation and Replacement schemata are the same. The two schemata are replaced with single axioms, using second-order quantifiers. Namely, Separation becomes the single axiom

$$\forall x\ \forall Y\ \exists z\ z = \{w \in x : w \in Y\}$$

and Replacement becomes the single axiom

$$\forall x\ \forall F\ (\forall a \exists! b\ (a, b) \in F) \Rightarrow \exists y\ y = F''x.$$

Formulating set theory in second-order logic possibly feels a bit uneasy. We're supposed to axiomatize what sets are, while using a logic that seems to presuppose the existence of sets? There may be some truth to that—it's a sticky philosophical issue, and I shan't wade into it—but we can talk with ZFC as our background theory, and the universe of sets as a setting. If we want to ask whether a transitive set M satisfies some second-order assertion, we take first-order variables to range over the elements of M while taking second-order variables to range over the subsets of M. There is no circularity with this approach.

Let us now characterize the inaccessible cardinals.

Theorem 25 (Zermelo). *Let M be a transitive set. Then $M \models \mathsf{ZFC}_2$ iff $M = V_\kappa$ for inaccessible κ.*

Proof. The backward direction is essentially the same arugment that $V_\kappa \models \mathsf{ZFC}$. (Exercise: finish the details!)

For the forward direction, suppose $M \models \mathsf{ZFC}_2$ is transitive. First, I claim that $M = V_\kappa$ for $\kappa = \mathrm{Ord} \cap M$. That $V_\kappa \subseteq M$ is proved by inductively showing that $V_\alpha \in M$ for all $\alpha < M$. The base case is trivial. The successor case follows from the second-order Separation axiom plus Powerset; if $V_\alpha \in M$ then $\mathcal{P}(V_\alpha) \in M$, because by second-order Separation M is correct about powersets. And the limit case follows by Replacement; if $V_\alpha \in M$ for all $\alpha < \lambda < \kappa$ then the sequence $\langle V_\alpha : \alpha < \lambda \rangle \in M$ and so $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$ is in M.

We must also see that $M \subseteq V_\kappa$. For this, it suffices to prove that if $x \in M$ then $\operatorname{rank} x < \kappa$. This follows from the usual ZFC argument that every set has a rank, which immediately applies in the second-order context.

Now let us see κ is an inaccessible cardinal. First, we will see κ is regular, and so in particular a cardinal. Suppose otherwise that there is $\alpha < \kappa$ and $G : \alpha \to \kappa$ with $\operatorname{ran} G$ cofinal in κ. Then, by second-order Replacement, $\operatorname{ran} G = G''\alpha \in V_\kappa$ and so $\sup G''\alpha = \kappa \in V_\kappa$, a contradiction. Now let us see κ is strong limit. If not, there would be $\lambda < \kappa$ so that $2^\lambda \geq \kappa$. But, because V_κ satisfies the Powerset axiom, we get that $\mathcal{P}(\lambda) \in V_\kappa$. So there is a surjection $G : \mathcal{P}(\lambda) \to \kappa$. By second-order Replacement, $G''\mathcal{P}(\lambda) = \kappa \in V_\kappa$, a contradiction. □

Zermelo adopted an upward dynamic view for the universe of sets. Quoting from his 1930 paper, as quoted in Kanamori's book.

> The ‘ultrafinite antinomies of set theory’ that scientific reactionaries and anti-mathematicians refer to so assiduously and lovingly in their campaign against set theory, these seeming ‘contradictions’, are only due to a confusion of set theory itself, which is non-categorically determined by its axioms, with particular representing models: What appears in one model as an ‘ultrafinite non-or metaset’ is in the next higher one already a fully valid ‘set’ with cardinal number and ordinal type, and is itself the foundation stone for the construction of the new domain. The unlimited series of Cantor’s ordinal numbers is matched by just as infinite a double series of essentially different set-theoretic models, the whole classical theory being manifested in each of them. The two diametrically opposite tendencies of the thinking spirit, the idea of creative progress and of comprehensive completion, which also lie at the root of the Kantian ‘antinomies’, find their symbolic representation and symbolic reconciliation in the transfinite series of numbers based on the concept of well-ordering. This series in its boundless progress does not have a true conclusion, only relative stopping points, namely those ‘limit numbers’ [inaccessible cardinals] which separate the higher from the lower model types. And thus also, the set-theoretic ‘antinomies’ lead, if properly understood, not to a restriction or mutilation but rather to a presently unserveyable unfolding and enrichment, of mathematical science.

Inaccessible cardinals can also be seen as embodying a certain reflection phenomenon. Recall that a reflection principle is a principle asserting that some V_α resembles V in some way. For example, the Lévy–Montague reflection principle, which is a theorem schema of ZFC, asserts that for any single first-order property V has there are lots and lots of V_αs which also have that property. The Powerset and Replacement axioms can be seen as asserting that Ord is inaccessible. Asserting the existence of inaccessible cardinals is saying that there are ordinals which resemble Ord in this way.

A natural question is then whether something like Zermelo’s theorem holds for first-order ZFC. We know that there are transitive models of ZFC which aren’t even V_κs, but maybe the only V_κs which satisfy all of ZFC are for κ inaccessible. This is not the case.

Definition 26. Say that a cardinal κ is *worldly* if $V_\kappa \models \mathsf{ZFC}$.

Theorem 27. *Suppose κ is inaccessible. Then κ is a limit of worldly cardinals.*

Proof. Fix $\alpha < \kappa$. We will see that there is $\lambda > \alpha$ so that $V_\lambda \prec V_\kappa$, which is enough to prove the result. This will be done by a variation of the Skolem hull argument used to prove the downward Löwenheim–Skolem theorem.

Fix Skolem functions for V_κ, which exist because V_κ can be well-ordered. Start with $\alpha_0 = \alpha$. Given α_n, let α_{n+1} be least so that $V_{\alpha_{n+1}}$ is closed under the Skolem functions with inputs from V_α. Finally, set $\lambda = \sup_n \alpha_n$. By a similar argument as before, $V_\lambda \subseteq V_\kappa$ satisfies the Tarski–Vaught test, so $V_\lambda \prec V_\kappa$. Finally, note that $\operatorname{cof} \lambda = \omega$ and so by the regularity of κ it must be that $\lambda < \kappa$. □

Exercise 28. Formulate a definition of inaccessible cardinals that works without assuming the axiom of choice. (The issue: if $\lambda < \kappa$ has that its powerset cannot be well-ordered, then we can’t possible have $2^\lambda < \kappa$.) Prove that inaccessible cardinals are limits of worldly cardinals without appealing to the axiom of choice. (But where κ is worldly if $V_\kappa \models \mathsf{ZF}$.)

4. Exercises

The first batch of exercises concern weakly inaccessible cardinals.

Definition 29. An uncountable cardinal κ is *weakly inaccessible* if κ is a regular limit cardinal.

Exercise 30. Assume GCH. Show that κ is inaccessible iff κ is weakly inaccessible.[11]

On the other hand, as we will see in part 2, it is consistent with ZFC that there are weakly inaccessible cardinals below $2^{\aleph_0}$.

Exercise 31. Analogously to the α-inaccessible cardinals, define the hierarchy of α-weakly inaccessible cardinals. Working from the assumption that there is an α-weakly inaccessible cardinal for every α, show that the smallest α-weakly inaccessible cardinal is not $(\alpha+1)$-weakly inaccessible.

The next exercises concern a connection to category theory.

Definition 32. A *Grothendieck universe* is a set U with the following properties.

- $\emptyset \in U$;
- U is transitive;
- If $x, y \in U$ then $\{x, y\} \in U$;
- If $x \in U$ then $\mathcal{P}(x) \in U$; and
- If $I \in U$ and $\{x_i : i \in I\} \in U$ then $\bigcup_{i \in I} x_i \in U$.

Grothendieck universes were used for a formalization of category theory. We want to be able to talk about things like the category of groups, or the category of sets, or so forth. But these are all proper classes. We can avoid proper classes by instead fixing a Grothendieck universe U and relativizing all the definitions to U. (And if we need to work higher and talk about U itself, than we just assume we have an even bigger Grothendieck universe.)

Exercise 33. Show that V_ω is a Grothendieck universe.

Exercise 34. Show that if κ is inaccessible then V_κ is a Grothendieck universe.

Exercise 35. Show that if U is an uncountable Grothendieck universe then $U = V_\kappa$ for some κ.

(Kameryn J. Williams) University of Hawaiʻi at Mānoa, Department of Mathematics, 2565 McCarthy Mall, Keller 401A, Honolulu, HI 96822, USA
E-mail address: kamerynw@hawaii.edu
URL: http://kamerynjw.net

[11]In fact, we can say more. If κ is weakly inaccessible then L_κ, the construction of Gödel's constructible universe up to κ, is a model of ZFC. But proving this fact would require talking about L, which we won't do in this class.